

SPRINGER BRIEFS IN
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Transient Heat Transfer

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1 Introduction

Let us consider Fig. 1 where the warm side of the wall is at temperature t_1 while the cold side is at temperature t_2 ; due to this temperature difference we know that a certain amount of heat transfers from the warm to the cold side.

Clearly, the heat entering the warm side is produced by an external source which transfers it to the wall; then the same heat is transferred from the cold wall to an external source.

Let us assume that the external source corresponding to the cold side suddenly transfers heat to it, thus increasing its temperature. For sake of simplicity, we assume that the temperature reaches t_1 , i.e., the identical temperature of the other side.

Clearly, at this point the heat transfer through the wall stops because the temperature difference between the two sides that caused it no longer exists.

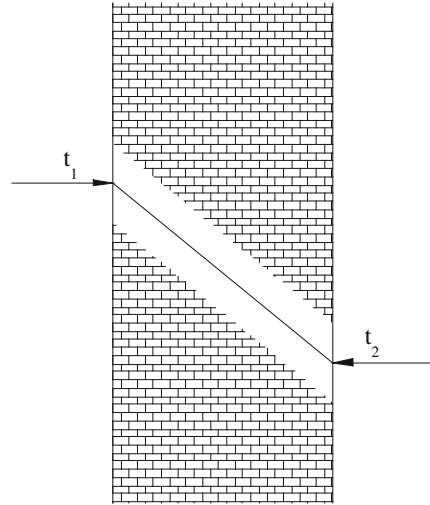
The temperature on the inside of the wall is lower compared to that of both sides, thus creating a heat flux towards the inside from both sides and progressively increasing the inside temperatures until the latter stabilize, depending on certain physical characteristics of the material. The example shows how the entire wall reaches temperature t_1 , and the heat transfer through the wall stops.

Naturally the phenomenon takes place in the same way, even though to a different degree, even when the temperature increase of the cold side is lower than the one assumed earlier. In that case, when balance is reached the heat transfer through the wall corresponds to the new difference in temperature between the sides.

Based on our assumptions and as a result of the unbalances in temperature, there is an increase in heat content of the wall.

Of course, the opposite takes place if the temperature of the warm wall suddenly drops. In that case there is a slow decrease in temperature inside the wall at the expense of a heat flux moving towards both sides until balance is reached, and the heat transfer through the wall is in agreement with the new temperatures of the sides.

Fig. 1 Wall with steady conduction



Evidently, if the temperature of the cold side is brought to be higher compared to the warm side, or if the temperature of the warm side is brought to be lower in comparison to the cold side, under steady conditions there is a change in heat transferring through the wall, and the direction of the heat flux is reversed.

As you can see, the situation shifts from initial balance of temperature to an unsteady state where temperatures vary over time, and finally to a new balance in agreement with the new temperatures of the sides.

If we consider a volume inside the wall that must receive heat from the adjacent areas during the unsteady state, this heat is proportional to the thermal conductivity of the material. The ensuing increase in temperature is proportional to the heat transfer, i.e., k , and is inversely proportional to the specific heat, as referred to the volume of the material. This specific heat (in $\text{J/m}^3 \text{ K}$) that we indicate with c_{vol} is given by the product of the specific heat referred to mass (in J/kg K) by the density (in kg/m^3), i.e.,

$$c_{\text{vol}} = c\rho. \quad (1)$$

Then we introduce thermal diffusivity a equal to

$$a = \frac{k}{c\rho}. \quad (2)$$

It is significant with regard to the duration of the transition.

The higher a , the faster the variations in temperature inside the wall with a consequent shorter duration of the unsteady state. Therefore, the latter is shorter in proportion to an increase of the value of k and a decrease in the value of product $c\rho$.

As we shall see, thermal diffusivity is included in Fourier's general law of thermal conduction.

2 General Law of Thermal Conduction

Let us consider the cubic element shown in Fig. 2 and assume that the element is crossed by heat only in the direction x .

If conditions are unsteady, heat dQ_1 enters the cubic element and heat dQ_2 exits the element. The latter differs from dQ_1 (under steady conditions both types of heat are, of course, identical); the heat dQ is stored in the cubic element and is equal to

$$dQ = dQ_1 - dQ_2. \quad (3)$$

If the left side of the element registers the thermal gradient $\partial t / \partial x$ (the derivative is partial given that t also depends on time), based on Fourier's well-known law, within time $d\theta$ and given that the surface crossed by the heat is equal to $dydz$, we have

$$dQ_1 = -k dydz \frac{\partial t}{\partial x} d\theta. \quad (4)$$

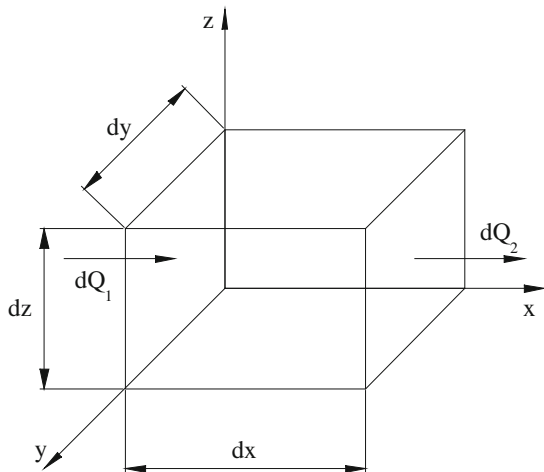
If we move to the other side of the cubic element, i.e., at distance dx from the previous one, the thermal gradient is given by

$$\frac{\partial t}{\partial x} + \frac{\partial}{\partial x} \frac{\partial t}{\partial x} dx = \frac{\partial t}{\partial x} + \frac{\partial^2 t}{\partial x^2} dx. \quad (5)$$

Therefore,

$$dQ_2 = -k dydz d\theta \left(\frac{\partial t}{\partial x} + \frac{\partial^2 t}{\partial x^2} dx \right). \quad (6)$$

Fig. 2 Element with transient conduction



Then, recalling (3)

$$dQ = k dy dz d\theta \frac{\partial^2 t}{\partial x^2} dx. \quad (7)$$

This heat increases the temperature of the cube; within time $d\theta$ this increase is equal to

$$\frac{\partial t}{\partial \theta} d\theta. \quad (8)$$

The heat dQ is equal to the volumetric specific heat of the material (given, as we already saw, by the product of the specific heat referred to mass c by the density of material ρ) multiplied by the volume of the cubic element and by the noted increase in temperature. Therefore,

$$dQ = c \rho dx dy dz \frac{\partial t}{\partial \theta} d\theta. \quad (9)$$

A comparison between (9) and (7) leads to

$$\frac{\partial t}{\partial \theta} = \frac{k}{c\rho} \frac{\partial^2 t}{\partial x^2}. \quad (10)$$

Equation 10 represents Fourier's general law of thermal conductivity.

As we can see, the partial derivative of temperature with respect to time goes up in sync with an increase in thermal diffusivity. Consequently, variations in temperature are faster and the duration of the unsteady state is reduced.

The integration of this differential equation makes it possible in principle to compute the development of temperature t over time for every point of the wall.

As shown, (10) was derived assuming that the heat transfer occurs only in one direction (x); this is the most frequent case, and from now on we shall always refer to this scenario. Please, note that if there is heat flux also in directions y and z the general law of thermal conduction changes as follows:

$$\frac{\partial t}{\partial \theta} = \frac{k}{c\rho} \left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \right). \quad (11)$$

Based on (10) we obtain rather complicated equations for the computation of t on which we shall not dwell; it suffices to notice that they are valid in principle but during an attempt to apply them to a specific case of interest, it is established that some of them do not satisfy the borderline conditions of space and time.

Therefore, it is required to use the equation or the equations that satisfy these conditions.

The analysis of the equations in question clearly shows that the approach to solve the issues of unsteady state is anything but easy.

Only few physically elementary situations can be studied in a relatively simple way.

For instance, in reference to a sudden variation of the superficial temperature only few types of variation find a theoretical solution.

For other situations that apparently look quite simple but already show complexity as far as the theoretical computation, the available literature provides complex and difficult to use theoretical treatises, solutions requiring the awkward and not necessarily precise use of diagrams, or approximate solutions.

The interest in such procedures faded after the introduction of automatic computation.

In fact, it is possible to substitute the resolution of differential equations with computation programs based on finite differences; we shall often exploit this possibility when examining the specific cases following next.

The use of finite differences had already been hypothesized before the advent of computers, but obviously the ensuing manual computation was enormously burdensome in terms of time; only the adoption of bigger dimensions of the differences made it possible to reasonably adopt the method, but in that case the errors were great, and the computation turned out to be rough. The introduction of automatic computation opened up possibilities that were previously unthinkable and eliminated all the problems mentioned above. By exploiting the enormous potential of automatic computation today, it is possible to examine even very complex cases and to adopt dimensions of the differences so minute to make the results basically exact.

Therefore, we believe it is neither useful nor advisable to illustrate the manual procedures described above.

By way of an example, we only illustrate a relatively simple type of theoretical computation.

3 Heat Diffusion in Plate of Infinite Thickness

We consider the side of an infinite thickness wall with a temperature equal to t_0 that is suddenly brought to temperature t_1 .

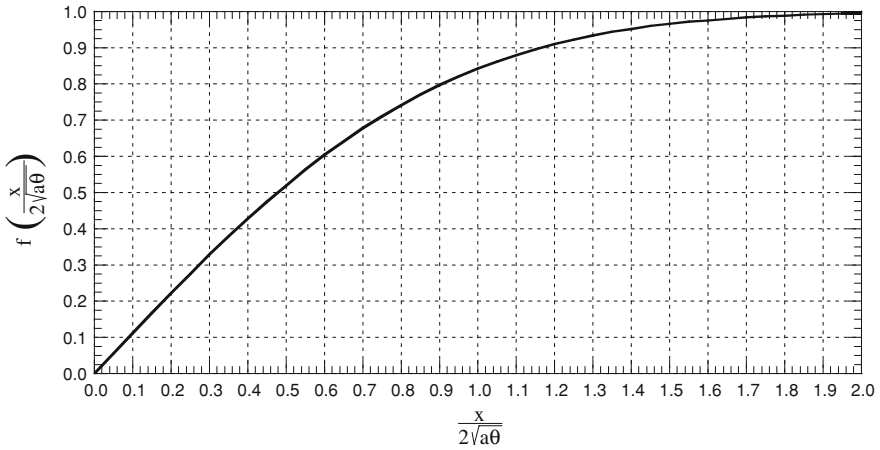
In other words, the aim is to establish how this sudden increase in temperature on the side impacts the temperature inside the wall over time.

The wall is assumed to have infinite thickness to allow the theoretical computation but, as we shall see, the results are transferable to a wall of finite thickness too, provided certain conditions are set.

From a practical point of view with regard to the investigation, note that the imaginary situation can be related to the ground when its surface is hit by a sudden increase in temperature; in that case the infinite thickness of the wall corresponds to reality.

Table 1 Error integral of Gauss

$\frac{x}{2\sqrt{a\theta}}$	$f()$	$\frac{x}{2\sqrt{a\theta}}$	$f()$	$\frac{x}{2\sqrt{a\theta}}$	$f()$	$\frac{x}{2\sqrt{a\theta}}$	$f()$
0.05	0.056	0.55	0.563	1.05	0.862	1.55	0.972
0.10	0.112	0.60	0.604	1.10	0.880	1.60	0.976
0.15	0.168	0.65	0.642	1.15	0.896	1.65	0.980
0.20	0.223	0.70	0.678	1.20	0.910	1.70	0.984
0.25	0.276	0.75	0.711	1.25	0.923	1.75	0.987
0.30	0.329	0.80	0.742	1.30	0.934	1.80	0.989
0.35	0.379	0.85	0.771	1.35	0.944	1.85	0.991
0.40	0.428	0.90	0.797	1.40	0.952	1.90	0.993
0.45	0.475	0.95	0.821	1.45	0.960	1.95	0.994
0.50	0.520	1.00	0.843	1.50	0.966	2.00	0.995

**Fig. 3** Function $f\left(\frac{x}{2\sqrt{a\theta}}\right)$

In this case the equation satisfying (10) is as follows:

$$t = A + Bx + C \frac{2}{\sqrt{\pi}} \int_{\eta=0}^{\eta=\frac{x}{2\sqrt{a\theta}}} e^{-\eta^2} d\eta. \quad (12)$$

Note that the integral in (12), i.e.,

$$\frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta \quad (13)$$

is none other than Gauss' *error integral*; its values may be taken from Table 1 or from Fig. 3; it is zero for $\eta = 0$ and it is equal to 1 for $\eta = \infty$.

At the beginning of the phenomenon, thus for $\theta = 0$, given that θ is time, and for $x = 0$, i.e., on the surface of the wall the value of η is zero, this means that the integral in question is zero.

Recalling (12) and the fact that the superficial temperature is increased to t_1 we have

$$t_1 = A. \quad (14)$$

Considering the generic section of the wall with abscissa x in the beginning of the phenomenon, i.e., for $\theta = 0$; this leads to $\eta = \infty$, and as a result the integral is equal to one.

Based on (12) and recalling that the wall is at temperature t_0 , we obtain

$$t = t_1 + Bx + C = t_0. \quad (15)$$

Equation 15 is in contrast with the fact that the entire wall is at temperature t_0 ; therefore, it is necessary to set $B = 0$; this leads to

$$C = t_0 - t_1. \quad (16)$$

Thus, based on (12)

$$t = t_1 + (t_0 - t_1)f\left(\frac{x}{2\sqrt{a\theta}}\right) \quad (17)$$

where $f(\cdot)$ indicates the error integral shown in Fig. 3; note that a stands for thermal diffusivity.

Thus, Eq. 17 makes it possible to compute the temperature in every section of the wall identified by the abscissa x and for any time indicated by θ .

It is interesting to establish the existing relationship between the variation in temperature occurring after a certain time in any location of the plate and the difference in temperature $(t_1 - t_0)$ between the temperature t_1 of the side and the initial one of the plate.

Based on (17)

$$\varphi = \frac{t - t_0}{t_1 - t_0} = 1 - f\left(\frac{x}{2\sqrt{a\theta}}\right). \quad (18)$$

As far as the phenomenon we are examining, Fourier's dimensionless number (Fo) is crucial; with reference to the generic position of the plate (thus considering distance x from the side brought to temperature t_1), it is given by

$$Fo = \frac{k\theta}{c\rho x^2} = \frac{a\theta}{x^2} \quad (19)$$

where a as usual represents thermal diffusivity.

From (18) and based on (19) we obtain

$$\varphi = 1 - \frac{1}{2\sqrt{Fo}}. \quad (20)$$

Fourier's number is clearly crucial as far as the value of φ .

The computation of heat crossing any surface identified by distance x , first of all requires the identification of the equation relative to the partial derivative $\partial t / \partial x$; it is given by

$$\frac{\partial t}{\partial x} = \frac{2e^{-\frac{x^2}{4a\theta}}}{\sqrt{\pi}} \frac{1}{2\sqrt{a\theta}} = \frac{e^{-\frac{x^2}{4a\theta}}}{\sqrt{a\pi\theta}}. \quad (21)$$

By recalling that with reference to the unit of time and surface, $q = -k\partial t / \partial x$. By deriving (17) we obtain

$$q = k(t_1 - t_0) \frac{e^{-\frac{x^2}{4a\theta}}}{\sqrt{a\pi\theta}}. \quad (22)$$

Per $x = 0$, thus in correspondence of the side of the plate, (22) is reduced to the following:

$$q = k \frac{t_1 - t_0}{\sqrt{a\pi\theta}}. \quad (23)$$

Now, if χ indicates the ratio between the heat crossing the surface identified by the distance x from the side at temperature t_1 and the heat crossing this side, based on (22) and (23) we obtain

$$\chi = e^{-\frac{x^2}{4a\theta}} \quad (24)$$

this can also be written as follows:

$$\chi = e^{-\frac{1}{4Fo}} \quad (25)$$

then

$$Fo = -\frac{1}{4 \log_e \chi}. \quad (26)$$

Please, note that any value of χ is matched by a value of Fourier's number; once this value is identified through (20) we obtain the corresponding value of φ .

This made Table 2 possible where any value of χ is matched by the corresponding values of Fo and φ .

Of course, once the value of Fo is known, the corresponding time θ is given by

$$\theta = Fo \frac{c\rho x^2}{k}. \quad (27)$$

4 Heat Transfer in Plate of Finite Thickness

In the beginning of the previous section we pointed out that Eq. 17 refers to a plate of infinite thickness; nonetheless, the above is valid also for a plate of finite thickness provided, of course, that the value of x is less than the thickness x_w of the plate, and that the heat crossing the other side is zero or irrelevant.

Table 2 Fourier's number and factor φ in function of χ

χ	Fo	φ	χ	Fo	φ
0.02	0.0639	0.0052	0.50	0.3607	0.2391
0.05	0.0834	0.0143	0.55	0.4182	0.2742
0.10	0.1086	0.0319	0.60	0.4894	0.3121
0.15	0.1318	0.0515	0.65	0.5803	0.3533
0.20	0.1553	0.0728	0.70	0.7009	0.3984
0.25	0.1803	0.0959	0.75	0.8690	0.4482
0.30	0.2076	0.1207	0.80	1.1203	0.5042
0.35	0.2381	0.1473	0.85	1.5383	0.5686
0.40	0.2728	0.1758	0.90	2.3728	0.6463
0.45	0.3131	0.2064	0.95	4.8739	0.7488

By considering this heat irrelevant, if it is equal to about 2% of the heat crossing the side at temperature t_1 , based on Table 2 we must obtain

$$x_w \geq 4\sqrt{\frac{k\theta}{c\rho}}. \quad (28)$$

Before proceeding with examples let us consider the following on Table 2.

Note that when the value of χ approaches unity the value of Fourier's number increases suddenly and considerably; for a given surface, thus setting the value of x the increase of Fo solely depends on the increase of time θ .

Thus, for the heat crossing the surface to be close to that hitting the plate through the side at temperature t_1 lots of time is required, even though as time goes by the heat hitting the plate continues to decrease, as shown in (23).

On the other hand, after looking at the values of φ , we establish that even when the heat crossing the surface is close to that hitting the plate, the local temperature of the plate is still far from temperature t_1 of the heated side.

4.1 Examples

- (1) Let us consider a steel plate at a given initial temperature; one of the sides is brought and kept at a higher temperature; for this type of analysis the value of these two temperatures is unimportant.

For steel we may assume that $c = 500 \text{ J/kg K}$; $\rho = 7850 \text{ kg/m}^3$; $k = 45 \text{ W/m K}$. The goal is to determine after how much time the heat passing at a distance of 60 mm from the heated side is equal to 60% of the heat hitting the plate.

Based on Table 2 for $\chi = 0.6$ we obtain $Fo = 0.4894$.

Thus, the required time is given by $\theta = 0.4894 \times 500 \times 7850 \times 0.06^2 / 45 = 153 \text{ s}$, i.e., about 2.5 min.

In order for the procedure to be valid (26) must be satisfied; so the thickness x_w of the plate must be $x_w \geq 4\sqrt{\frac{45 \times 153}{500 \times 7850}} = 0.167 \text{ m} = 167 \text{ mm}$.

Finally, based on Table 2 $\varphi = 0.312$; for instance, if $t_0 = 20^\circ\text{C}$ and $t_1 = 200^\circ\text{C}$, we would have $(t - 20)/(200 - 20) = 0.312$ and then $t = 76^\circ\text{C}$.

Therefore, in the indicated position the plate would be at 76°C .

In this case, having assumed that $x_w = 180$ mm, the behavior of χ and of the temperature are shown in Fig. 4.

- (2) Let us consider a concrete plate where one side is brought and kept at a higher temperature compared to the initial temperature of the plate.

Given that $c = 880$ J/kg K; $\rho = 2300$ kg/m³; $k = 0.8$ W/m K.

The goal is to establish after how much time, considering a distance of 100 mm from the heated side, the heat passing through is equal to 50% of the one hitting the plate.

Based on Table 2 for $\chi = 0.5$ $Fo = 0.3607$; thus, the required time is equal to $\theta = 0.3607 \times 880 \times 2300 \times 0.1^2/0.8 = 9126$ s; this corresponds to 152 min, i.e., to about 2.5 h.

In order for the procedure to be valid (28) must be satisfied, so the thickness of the plate must be $x_w \geq 4\sqrt{\frac{0.8 \times 9126}{880 \times 2300}} = 0.240$ m = 240 mm.

As far as local temperature t , based on Table 2, $\varphi = 0.2391$.

As in the previous case, assuming that $t_0 = 20^\circ\text{C}$ and $t_1 = 200^\circ\text{C}$ we reach $(t - 20)/(200 - 20) = 0.2391$ and then $t = 63^\circ\text{C}$.

In this case, having assumed that $x_w = 240$ mm, the behavior of χ and of the temperature is shown in Fig. 5.

Fig. 4 Steel plate—
thickness 180 mm

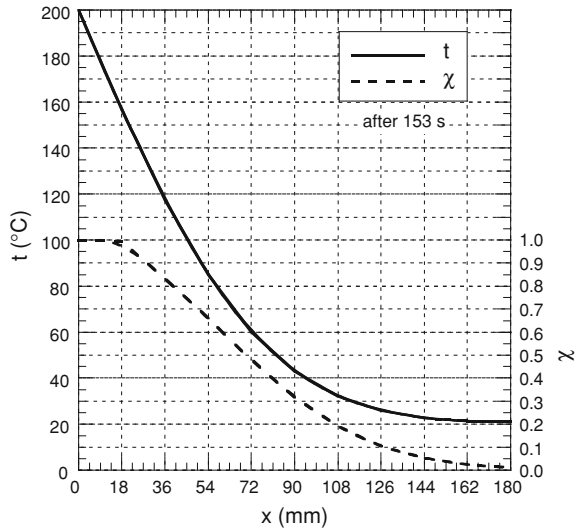
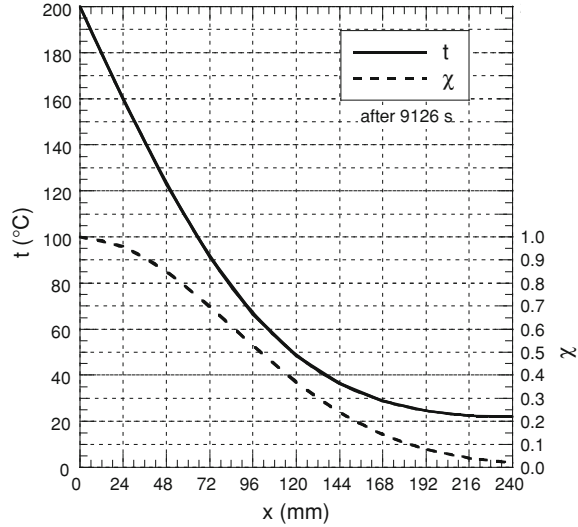


Fig. 5 Concrete plate—
thickness 240 mm



5 Plate Immersed in Fluid

As stated in the introduction, in this case as well as in the following ones, we shall use a finite differences program.

Let us consider a flat plate (e.g., a steel plate), at a given initial temperature, which is immersed in a fluid at a different temperature that either increases or decreases its temperature; we know that the plate reaches the temperature of the fluid in a theoretically infinite time.

On the other hand, if we conventionally assume that this process may practically be considered complete when the difference in temperature between fluid and plate is reduced to 5% of its initial one (in other words, the increase or decrease in temperature is equal to 95% of the initial difference in temperature) it is possible to identify the required time.

The time in question indicated by θ , is a function of various quantities, specifically: the thickness x_w of the plate, the specific heat c , the density ρ , the thermal conductivity k of the material of the plate, and finally the heat transfer coefficient α of the fluid surrounding the plate.

We know that for the transient phenomena we are discussing the dimensionless number of Fourier is fundamental. By indicating it with Fo , it is given by

$$Fo = \frac{k}{c\rho x_w^2}\theta. \quad (29)$$

By analyzing the condition described above, i.e., the increase or decrease in temperature equal to 95%, we see how the various quantities in question influence the value of both θ and Fo .

We establish that the value of θ is proportional to the value of c and ρ ; the specific heat and the density of the material do not impact the value of Fourier's number.

A decrease of the value of k increases the value of θ but this increase is generally modest; in any case it is far from proportionality between θ and the reciprocal of k ; thus, the value of Fo decreases with a decrease of the value of k .

If the thickness of the plate increases the value of θ goes up but the increase is not such to lead to proportionality between θ and the square of x_w ; so if x_w increases, Fo decreases.

Finally, if we consider the heat transfer coefficient α of the surrounding fluid which is not included in Fourier's number, it is obvious that an increase of α causes the value of θ (and consequently of Fo) to decrease; in fact, the increase of α facilitates the heat transfer from the fluid to the plate and viceversa.

Therefore, we establish that Fo decreases if α and x_w increase, and if k decreases.

At this point it becomes natural to consider Biot's dimensionless number indicated by Bi , and given by

$$Bi = \frac{\alpha x_w}{k}. \quad (30)$$

Please, note that Biot's number is formally identical to Nusselt's number; the difference consists in the fact that Nusselt's number considers the values of α and k in reference to a fluid and considers the diameter of the tube where the fluid flows instead of the thickness x_w ; this is why this number is fundamental in the analysis of heat transfer by convection from a fluid to the wall of a tube; on the other hand, while Biot's number considers the heat transfer coefficient α of the fluid, it also analyzes the thermal conductivity of the material of the solid and its thickness; these are in fact the quantities of interest to us.

By analyzing the values of θ and Fo in reference to variations in the quantities in question, we establish that the value of Fo univocally depends on the value of Bi ; in other words:

$$Fo = f(Bi). \quad (31)$$

The values of Fo as a function of Bi for increase or decrease in temperature of the plate of 95% are listed in the last column in Table 3.

Note that the increase or decrease in temperature of the plate may also be partial; in particular, it is possible to use a fluid at a higher temperature compared to the final temperature desired for the plate.

Thus, it is interesting to be able to have the values of Fo as a function of Bi even in these cases.

Table 3 shows the values of Fo for a temperature increase or decrease equal to 30, 40, 50, 60, 70, 80 and 90% of the initial temperature difference between fluid and plate.

Note that the final temperature of the plate is the average one across the thickness; it is important not to forget that the temperature of the plate varies across the thickness depending on the circumstances; in the case of temperature

Table 3 Fourier's number (Fo) for plate immersed in fluid

Bi	Plate heating or cooling (%)							
	30	40	50	60	70	80	90	95
0.010	17.51	25.08	34.03	44.75	58.80	77.94	111.49	142.6
0.015	11.68	16.73	22.70	30.01	39.44	52.45	74.39	96.80
0.020	8.767	12.56	17.04	22.53	29.60	39.57	56.32	72.69
0.025	7.020	10.06	13.64	18.04	23.71	31.69	45.10	58.68
0.030	5.854	8.385	11.38	15.04	19.77	26.43	37.80	48.95
0.04	4.398	6.299	8.548	11.30	14.85	19.86	28.40	36.95
0.05	3.523	5.047	6.850	9.055	11.90	15.91	22.76	29.62
0.06	2.941	4.213	5.718	7.559	9.934	13.28	19.00	24.73
0.07	2.525	3.617	4.909	6.490	8.528	11.40	16.31	21.23
0.08	2.212	3.170	4.302	5.688	7.475	9.994	14.30	18.61
0.10	1.776	2.544	3.453	4.565	6.000	8.021	11.48	14.94
0.12	1.484	2.126	2.887	3.818	5.017	6.707	9.596	12.49
0.15	1.193	1.710	2.321	3.069	4.033	5.393	7.716	10.04
0.20	0.902	1.293	1.755	2.321	3.050	4.078	5.836	7.593
0.25	0.727	1.042	1.415	1.872	2.460	3.289	4.708	6.126
0.30	0.610	0.875	1.189	1.572	2.067	2.764	3.955	5.147
0.40	0.4641	0.666	0.905	1.198	1.576	2.107	3.015	3.924
0.50	0.3765	0.541	0.736	0.973	1.280	1.713	2.451	3.191
0.60	0.3181	0.4574	0.622	0.824	1.084	1.450	2.076	2.702
0.80	0.2450	0.3528	0.4803	0.637	0.838	1.121	1.606	2.091
1.0	0.2010	0.2900	0.3953	0.524	0.690	0.924	1.324	1.725
1.2	0.1716	0.2480	0.3385	0.4492	0.592	0.793	1.137	1.481
1.4	0.1505	0.2180	0.2979	0.3956	0.522	0.699	1.003	1.307
1.7	0.1281	0.1862	0.2548	0.3389	0.4473	0.600	0.861	1.123
2.0	0.1122	0.1638	0.2246	0.2992	0.3953	0.531	0.762	0.994
2.5	0.0942	0.1382	0.1904	0.2541	0.3363	0.4523	0.650	0.848
3.0	0.0820	0.1211	0.1673	0.2240	0.2971	0.4000	0.576	0.752
4.0	0.0667	0.0994	0.1385	0.1863	0.2479	0.3347	0.4832	0.632
5.0	0.0573	0.0863	0.1210	0.1636	0.2184	0.2956	0.4278	0.560
6.0	0.0509	0.0775	0.1093	0.1484	0.1987	0.2697	0.3910	0.513
8.0	0.0428	0.0662	0.0945	0.1293	0.1741	0.2373	0.3453	0.4534
10.0	0.0379	0.0594	0.0855	0.1177	0.1593	0.2179	0.3181	0.4183

increase of the plate the temperature in the middle is lower compared to that on the sides of the plate; in the case of decrease of temperature the temperature in the middle is higher than the one on the sides.

At this point the analysis must go further.

We believe, in fact, that instead of referring to Fourier's number it is more appropriate to refer to a new dimensionless number.

It is the product of Fourier's number by Biot's number.

By indicating it with X , it is therefore equal to

$$X = Fo \times Bi = \frac{\alpha \theta}{c \rho x_w}. \quad (32)$$

Please note that in comparison with Fo this dimensionless number does not include k ; α is included instead, and the thickness x_w is no longer squared.

The values of X as a function of Bi and of the percentage of temperature increase or decrease are shown in Table 4.

The analysis of the values of X leads to interesting considerations.

Note that for values of Bi ranging from 0.01 and 0.1 X is practically independent from Bi ; this means that the influence of k on θ is irrelevant and that the time θ is about inversely proportional to α and about proportional to thickness x_w .

Table 4 Dimensionless number X for plate immersed in fluid

Bi	Plate heating or cooling (%)								
	20	30	40	50	60	70	80	90	95
0.010	0.1095	0.1751	0.2507	0.3402	0.4475	0.5880	0.7793	1.1149	1.4260
0.015	0.1096	0.1752	0.2509	0.3405	0.4501	0.5915	0.7866	1.1158	1.4520
0.020	0.1097	0.1753	0.2511	0.3408	0.4505	0.5920	0.7914	1.1264	1.4538
0.025	0.1098	0.1755	0.2513	0.3410	0.4509	0.5925	0.7920	1.1275	1.4670
0.030	0.1098	0.1756	0.2515	0.3414	0.4512	0.5930	0.7927	1.1340	1.4685
0.040	0.1100	0.1759	0.2520	0.3419	0.4520	0.5940	0.7940	1.1360	1.4780
0.050	0.1102	0.1762	0.2524	0.3425	0.4528	0.5950	0.7954	1.1380	1.4810
0.060	0.1103	0.1764	0.2528	0.3431	0.4535	0.5960	0.7967	1.1400	1.4838
0.070	0.1105	0.1767	0.2532	0.3436	0.4543	0.5970	0.7980	1.1417	1.4861
0.080	0.1107	0.1770	0.2536	0.3442	0.4550	0.5980	0.7995	1.1440	1.4888
0.10	0.1110	0.1776	0.2544	0.3453	0.4565	0.6000	0.8021	1.1480	1.4940
0.12	0.1113	0.1781	0.2552	0.3464	0.4581	0.6020	0.8048	1.1515	1.4988
0.15	0.1118	0.1789	0.2564	0.3481	0.4603	0.6049	0.8089	1.1574	1.5060
0.20	0.1126	0.1803	0.2584	0.3509	0.4641	0.6100	0.8156	1.1672	1.5186
0.25	0.1133	0.1817	0.2605	0.3537	0.4679	0.6150	0.8223	1.1770	1.5315
0.30	0.1141	0.1830	0.2625	0.3565	0.4716	0.6200	0.8291	1.1865	1.5441
0.40	0.1156	0.1856	0.2664	0.3621	0.4791	0.6300	0.8426	1.2060	1.5696
0.50	0.1171	0.1883	0.2705	0.3677	0.4865	0.6400	0.8561	1.2255	1.5955
0.60	0.1185	0.1909	0.2744	0.3732	0.4941	0.6500	0.8697	1.2456	1.6212
0.80	0.1213	0.1960	0.2822	0.3842	0.5091	0.6701	0.8968	1.2848	1.6728
1.0	0.1241	0.2010	0.2900	0.3952	0.5240	0.6902	0.9242	1.3240	1.7250
1.2	0.1264	0.2059	0.2976	0.4062	0.5390	0.7102	0.9516	1.3644	1.7772
1.4	0.1288	0.2106	0.3052	0.4170	0.5538	0.7303	0.9790	1.4042	1.8298
1.7	0.1323	0.2176	0.3164	0.4332	0.5761	0.7604	1.0202	1.4637	1.9091
2.0	0.1356	0.2245	0.3274	0.4492	0.5984	0.7906	1.0616	1.5240	1.9880
2.5	0.1408	0.2355	0.3455	0.4758	0.6352	0.8409	1.1306	1.6250	2.1200
3.0	0.1458	0.2461	0.3632	0.5020	0.6720	0.8911	1.1999	1.7280	2.2560
4.0	0.1552	0.2665	0.3977	0.5539	0.7450	0.9915	1.3388	1.9328	2.5280
5.0	0.1640	0.2862	0.4316	0.6050	0.8176	1.0918	1.4783	2.1390	2.8000
6.0	0.1725	0.3053	0.4647	0.6556	0.8899	1.1921	1.6180	2.3460	3.0780
8.0	0.1888	0.3424	0.5298	0.7558	1.0337	1.3924	1.8980	2.7624	3.6272
10.0	0.2045	0.3785	0.5937	0.8548	1.1768	1.5926	2.1786	3.1810	4.1830

For higher values of Bi and specifically for extremely high values of Bi the impact of k on θ starts to be noticeable, the impact of α on θ decreases while the impact of x_w on θ creases.

In fact, if we consider the values of X shown in the table for $Bi > 1$:

$$X = X_1 Bi^\beta \quad (33)$$

where X_1 represents the value of X for $Bi = 1$ and exponent β varies for all the examined instances between 0.16 and 0.38.

Therefore,

$$\frac{\alpha\theta}{c\rho x_w} = X_1 \left(\frac{\alpha x_w}{k} \right)^\beta \quad (34)$$

then

$$\theta \equiv \frac{x_w^{1+\beta}}{\alpha^{1-\beta} k^\beta}. \quad (35)$$

This establishes that the exponent of x_w is greater than one, whereas the exponent of α is smaller than one, to the contrary of what happens for values of Bi ranging from 0.01 and 0.1; finally, this highlights the impact of k on θ .

This greater influence of k on θ is due to the fact that, as we shall see later on, for the high values of Bi the difference between the temperature of the sides of the plate and the mean value of the temperature across the thickness is more highlighted, difference which is obviously influenced by the value of k .

Once the value of X is known through Table 4 based on the value of Bi , the time θ required to obtain the desired result is computed through (32), given that

$$\theta = X \frac{c\rho x_w}{\alpha}. \quad (36)$$

Clearly, the computation is particularly easy and it is possible to avoid using a finite differences program each time that would be necessary to solve this task.

At this point let us look at a few examples.

5.1 Examples

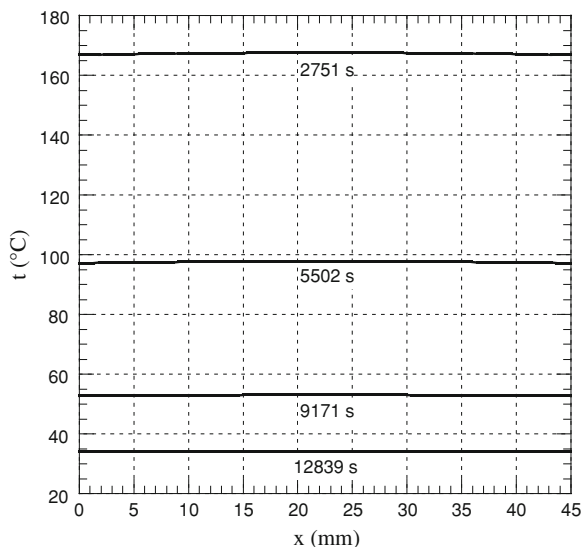
- (1) Let us consider a steel plate with a thickness of 45 mm, initially at a temperature of 300°C which is to be cooled through air to a temperature of 20°C. Considering that $c = 500$ J/kg K; $\rho = 7850$ kg/m³; $k = 45$ W/m K; $\alpha = 20$ W/m² K.

Biot's number is therefore equal to $20 \times 0.045/45 = 0.02$.

If we consider a decrease in temperature equal to 95%, i.e., a final temperature of the metal plate equal to 34°C, based on Table 4 we obtain $X = 1.4538$.

Based on (36) we have $\theta = 1.4538 \times 500 \times 7850 \times 0.045/20 = 12839$ s corresponding to 214 min., thus about 3 and a half hours.

Fig. 6 Steel plate cooled on both sides—thickness 45 mm



Moreover, note that based on the values of X , the steel plate goes from 300 to 216°C in 1548 s, to 160°C in 3010 s and to 104°C in 5228 s.

The behavior of the temperatures through the wall and depending on time variations is shown in Fig. 6.

- (2) Let us now consider a steel plate with a thickness of 90 mm at an initial temperature of 20°C to be heated to 300°C through a fluid at 420°C. Considering that $c = 500 \text{ J/kg K}$; $\rho = 7850 \text{ kg/m}^3$; $k = 45 \text{ W/m K}$; $\alpha = 200 \text{ W/m}^2 \text{ K}$.

Biot's number is equal to $200 \times 0.09 \times 45 = 0.4$.

The heating is equal to $280/400 = 0.7 = 70\%$.

Based on Table 4 $X = 0.6304$.

Therefore, based on (36) $\theta = 0.6304 \times 500 \times 7850 \times 0.09/200 = 1113 \text{ s}$, corresponding to about 18 min.

In addition, note that based on the values of X , the plate goes from 20 to 140°C in 328 s and to 220°C in 639 s.

If the plate were to be heated to 400°C, 2771 s would be required, and that would correspond to about 46 min.

The behavior of the temperatures through the wall and depending on time variations is shown in Fig. 7.

- (3) Let us consider a plate made of refractory material with a thickness of 75 mm, and a temperature of 500°C to be reduced through air to 20°C.

Considering that $c = 1200 \text{ J/kg K}$; $\rho = 2000 \text{ kg/m}^3$; $k = 1.5 \text{ W/m K}$; $\alpha = 20 \text{ W/m}^2 \text{ K}$.

Biot's number is equal to $20 \times 0.075/1.5 = 1$.

Let us also consider cooling equal to 95% so that the plate reaches a temperature of 44°C.

Fig. 7 Steel plate heated on both sides—thickness 90 mm

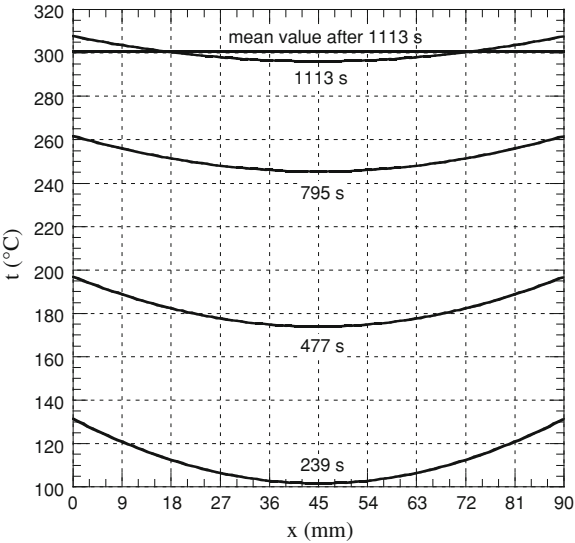
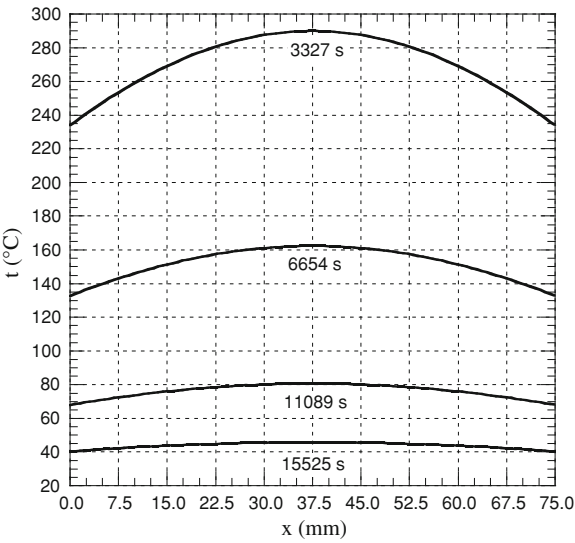


Fig. 8 Refractory plate cooled on both sides—thickness 75 mm



Based on Table 4 we obtain $X = 1.725$.
Therefore, $\theta = 1.725 \times 1200 \times 2000 \times 0.075/20 = 15525$ s, equal to 258 min and about 4.3 h.
Note that the temperature of the plate decreases to 308°C after 2610 s (43.5 min), to 212°C after 4716 s (78.6 min) and to 116°C after 8318 s (138.6 min); this establishes that going from 116 to 44°C requires 2 h.
The behavior of the temperatures through the wall and depending on variations in time is shown in Fig. 8.

- (4) Assuming at this point that the same plate of the previous example is to be heated from 20 to 160°C with a fluid at 300°C with a heat transfer coefficient of 100 W/m² K.

Heating is equal to $140/280 = 0.5 = 50\%$.

Biot's number is equal to $100 \times 0.075/1.5 = 5$.

Based on Table 4, $X = 0.605$.

Thus, $\theta = 0.605 \times 1200 \times 2000 \times 0.075/100 = 1089$ s, equal to about 18 min.

The behavior of the temperatures through the wall and depending on time variations is shown in Fig. 9.

6 Plate Heated or Cooled on One Side

Even in this case the analysis of the impact of the various quantities on the time required to obtain a certain result shows that the value of Fourier's number solely depends on Biot's number.

In view of the previous section we will not compute Fourier's number as a function of Bi and analyze the values of the dimensionless number X instead, given that we consider it more significant for our study.

Let us examine a plate licked by a fluid on just one side either receiving heat from the fluid or transferring heat to it assuming either no heat loss through the other side or a negligible one.

Under these assumptions it was possible to build Table 5 listing the values of dimensionless number X as a function of Biot's number (Bi).

Fig. 9 Refractory plate heated on both sides—thickness 75 mm

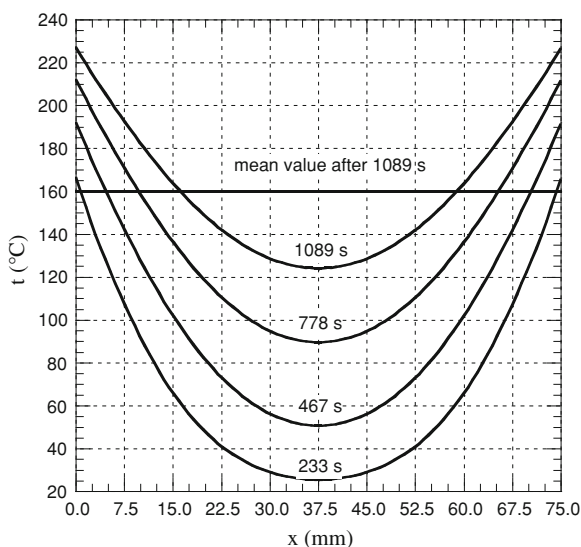


Table 5 Dimensionless number X for plate heated or cooled on one side

Bi	Plate heating or cooling (%)						
	20	30	40	50	60	70	80
0.010	0.2194	0.3507	0.4997	0.6723	0.8888	1.1612	1.5343
0.015	0.2198	0.3513	0.5031	0.6792	0.8979	1.1698	1.5637
0.020	0.2201	0.3519	0.5040	0.6838	0.8994	1.1818	1.5663
0.025	0.2205	0.3524	0.5048	0.6850	0.9055	1.1838	1.5824
0.030	0.2208	0.3530	0.5056	0.6861	0.9070	1.1918	1.5851
0.040	0.2215	0.3541	0.5073	0.6884	0.9100	1.1957	1.5986
0.050	0.2222	0.3553	0.5089	0.6906	0.9130	1.1997	1.6038
0.060	0.2229	0.3564	0.5106	0.6929	0.9160	1.2036	1.6091
0.070	0.2236	0.3576	0.5122	0.6951	0.9190	1.2076	1.6145
0.080	0.2243	0.3587	0.5139	0.6974	0.9220	1.2115	1.6197
0.10	0.2256	0.3609	0.5172	0.7019	0.9280	1.2195	1.6303
0.12	0.2270	0.3632	0.5204	0.7064	0.9340	1.2274	1.6410
0.15	0.2290	0.3665	0.5253	0.7131	0.9429	1.2393	1.6569
0.20	0.2323	0.3721	0.5335	0.7243	0.9579	1.2591	1.6835
0.25	0.2355	0.3775	0.5415	0.7355	0.9728	1.2789	1.7101
0.30	0.2386	0.3829	0.5495	0.7466	0.9878	1.2987	1.7369
0.40	0.2447	0.3936	0.5655	0.7688	1.0176	1.3384	1.7905
0.50	0.2505	0.4040	0.5812	0.7908	1.0473	1.3781	1.8443
0.60	0.2561	0.4142	0.5968	0.8128	1.0771	1.4179	1.8982
0.80	0.2666	0.4339	0.6274	0.8563	1.1363	1.4974	2.0064
1.0	0.2765	0.4530	0.6574	0.8993	1.1954	1.5770	2.1150
1.2	0.2858	0.4715	0.6870	0.9420	1.2541	1.6566	2.2238
1.4	0.2948	0.4895	0.7160	0.9843	1.3128	1.7362	2.3330
1.7	0.3076	0.5157	0.7589	1.0473	1.4004	1.8556	2.4974
2.0	0.3200	0.5411	0.8010	1.1096	1.4876	1.9750	2.6620
2.5	0.3396	0.5822	0.8697	1.2123	1.6323	2.1738	2.9372
3.0	0.3584	0.6221	0.9373	1.3140	1.7763	2.3726	3.2131
4.0	0.3944	0.6995	1.0698	1.5152	2.0629	2.7697	3.7660
5.0	0.4290	0.7748	1.1999	1.7145	2.3483	3.1666	4.3204
6.0	0.4627	0.8487	1.3286	1.9124	2.6329	3.5634	4.8749
8.0	0.5283	0.9941	1.5833	2.3059	3.2005	4.3563	—
10.0	0.5925	1.1376	1.8358	2.6977	3.7669	—	—

We limited the analysis to maximum heating or cooling equal to 80%, given that the expectation of more heating or cooling licking the plate on just one side is unlikely.

The values of X are of course higher than those corresponding to a plate immersed in fluid (Table 4); heating or cooling through one side only obviously requires longer time, thus increasing the value of X .

The impact of k on θ increases, too, given the more difficult heat transfer from the only involved side towards the inside the plate.

Specifically, we establish the following.

Within the values of Bi ranging between 0.01 and 0.1 the impact of k is still modest if not irrelevant, as in the case of the plate immersed in fluid.

The analysis of the values of X in this area establishes that with reference to (35) the exponent β is never greater than 0.026; the impact of k on θ remains modest.

If we examine the values of X for $Bi = 0.1$ – 1 the exponent β varies from 0.034 to 0.113.

Finally, if we look at the values of X for $Bi = 1$ – 10 the exponent β varies from 0.18 to 0.5.

Even in this case it is best to cite some examples.

For a useful comparison with the plate immersed in fluid we consider two examples already discussed in the previous section.

6.1 Examples

- (1) Let us consider a steel plate with a thickness of 90 mm, initially at 20°C to be heated to 300°C through a fluid at 420°C licking only one side.

Considering that $c = 500 \text{ J/kg K}$; $\rho = 7850 \text{ kg/m}^3$; $k = 45 \text{ W/m K}$; $\alpha = 200 \text{ W/m}^2 \text{ K}$.

Biot's number is equal to $200 \times 0.09/45 = 0.4$.

The heating is equal to $280/400 = 0.7 = 70\%$.

Based on Table 5, $X = 1.3384$.

Thus, $\theta = 1.3384 \times 500 \times 7850 \times 0.09/200 = 2364 \text{ s}$, corresponding to about 39 min.

Comparing this result with that of the plate immersed in fluid we establish that with the fluid licking the plate only on one side the required time more than doubles.

The behavior of the temperatures for various amounts of time including, of course, 2364 s, is shown in Fig. 10.

- (2) Let us assume a plate made of refractory material with a thickness of 75 mm, initially at 20°C to be heated to 160°C through a fluid at 300°C licking only on one side.

Considering that $c = 1200 \text{ J/kg K}$; $\rho = 2000 \text{ kg/m}^3$; $k = 1.5 \text{ W/m K}$; $\alpha = 100 \text{ W/m}^2 \text{ K}$.

Biot's number is equal to $100 \times 0.075/1.5 = 5$.

Heating is equal to $140/280 = 0.5 = 50\%$.

Based on Table 5, $X = 1.7145$.

Therefore, $\theta = 1.7145 \times 1200 \times 2000 \times 0.075/100 = 3086 \text{ s}$, corresponding to about 51 min.

A comparison of the value obtained with the one corresponding to a plate immersed in fluid shows that the required time is almost triple.

The behavior of the temperatures for various amounts of time, such as 3086 s, is shown in Fig. 11.

Fig. 10 Steel plate heated on one side—thickness 90 mm

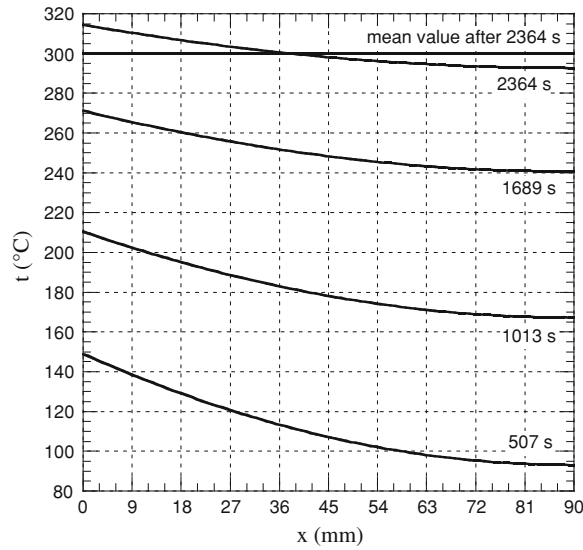
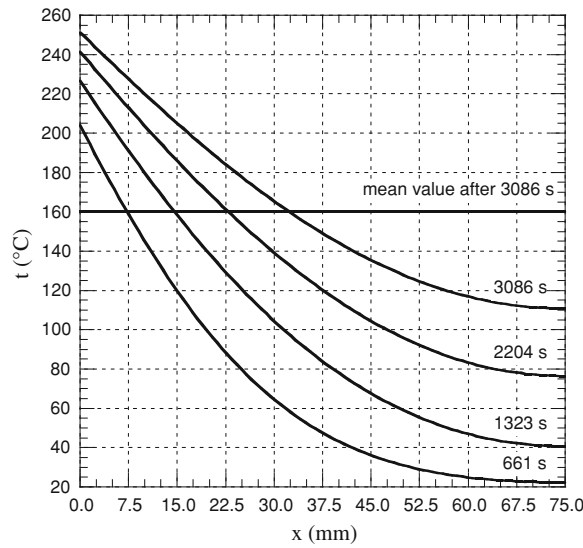


Fig. 11 Refractory plate heated on one side—thickness 75 mm



7 Tube Heated or Cooled on Outside Surface

In this section we examine a tube heated or cooled through a fluid licking it on the outside surface, while the heat transfer to the internal fluid through the inside surface either does not take place or is negligible.

This case is similar to the one discussed in the previous section regarding a plate licked by a fluid on just one side.

Regardless of their similarity the two cases differ due to the curvature of the wall of the tube; thus, given equal thickness and equal values for the other quantities they produce different values of the dimensionless number X .

As we know, time θ which is necessary to cause a certain event, depends on the value of X ; therefore, given equal conditions, timing for the tube is different compared to the plate.

By first approximation (but in almost all instances the obtained results are very close to reality) it is possible to use the following equation:

$$X = X_p \left(1 - \frac{x_w}{D_o} \right). \quad (37)$$

In (37) X_p is the value of X relative to the plate (Table 5), x_w is the thickness of the wall of the tube, and D_o is the outside diameter of the tube.

In the case of a bar instead, where $x_w/D_o = 0.5$, based on (37) we obtain $X = 0.5X_p$; in one of the following examples we will discuss this case, as well.

To facilitate reading Table 6 lists the values of the dimensionless number X for $x_w/D_o = 0.1$, and Table 7 lists the values for X per $x_w/D_o = 0.2$.

Table 6 Dimensionless number X for tube heated or cooled on outside surface ($x_w/D_o = 0.1$)

Bi	Tube heating or cooling (%)						
	20	30	40	50	60	70	80
0.010	0.1980	0.3165	0.4509	0.6116	0.8021	1.0522	1.3847
0.015	0.1984	0.3171	0.4541	0.6154	0.8104	1.0558	1.4112
0.020	0.1986	0.3176	0.4549	0.6171	0.8150	1.0666	1.4125
0.025	0.1991	0.3183	0.4550	0.6174	0.8162	1.0670	1.4263
0.030	0.1994	0.3188	0.4566	0.6184	0.8175	1.0742	1.4287
0.040	0.2000	0.3198	0.4582	0.6205	0.8202	1.0777	1.4409
0.050	0.2007	0.3209	0.4596	0.6225	0.8246	1.0814	1.4456
0.060	0.2013	0.3219	0.4612	0.6245	0.8273	1.0870	1.4504
0.070	0.2023	0.3230	0.4626	0.6278	0.8300	1.0907	1.4552
0.080	0.2030	0.3240	0.4641	0.6299	0.8327	1.0942	1.4599
0.10	0.2042	0.3259	0.4671	0.6339	0.8381	1.1014	1.4724
0.12	0.2054	0.3287	0.4700	0.6380	0.8435	1.1085	1.4821
0.15	0.2077	0.3317	0.4744	0.6440	0.8516	1.1193	1.4964
0.20	0.2111	0.3367	0.4828	0.6542	0.8651	1.1372	1.5205
0.25	0.2144	0.3423	0.4900	0.6656	0.8786	1.1550	1.5445
0.30	0.2177	0.3479	0.4983	0.6756	0.8921	1.1729	1.5687
0.40	0.2237	0.3576	0.5128	0.6957	0.9209	1.2112	1.6171
0.50	0.2294	0.3678	0.5281	0.7171	0.9478	1.2471	1.6690
0.60	0.2355	0.3778	0.5422	0.7370	0.9767	1.2831	1.7178
0.80	0.2461	0.3974	0.5723	0.7780	1.0304	1.3578	1.8157
1.0	0.2563	0.4157	0.5997	0.8187	1.0861	1.4300	1.9178

Table 7 Dimensionless number X for tube heated or cooled on outside surface ($x_w/D_o = 0.2$)

Bi	Tube heating or cooling (%)						
	20	30	40	50	60	70	80
0.010	0.1760	0.2810	0.4023	0.5435	0.7120	0.9359	1.2295
0.015	0.1763	0.2817	0.4035	0.5469	0.7193	0.9456	1.2531
0.020	0.1769	0.2822	0.4042	0.5478	0.7241	0.9468	1.2582
0.025	0.1772	0.2829	0.4048	0.5488	0.7254	0.9531	1.2680
0.030	0.1773	0.2834	0.4055	0.5502	0.7274	0.9548	1.2702
0.040	0.1780	0.2845	0.4068	0.5521	0.7298	0.9589	1.2810
0.050	0.1789	0.2855	0.4085	0.5538	0.7322	0.9621	1.2862
0.060	0.1795	0.2864	0.4099	0.5557	0.7346	0.9652	1.2894
0.070	0.1804	0.2876	0.4112	0.5580	0.7370	0.9684	1.2938
0.080	0.1810	0.2885	0.4129	0.5598	0.7394	0.9715	1.2992
0.10	0.1824	0.2906	0.4156	0.5634	0.7449	0.9780	1.3077
0.12	0.1839	0.2927	0.4186	0.5676	0.7498	0.9853	1.3163
0.15	0.1859	0.2960	0.4229	0.5730	0.7577	0.9948	1.3291
0.20	0.1893	0.3011	0.4304	0.5832	0.7697	1.0107	1.3518
0.25	0.1927	0.3061	0.4373	0.5928	0.7825	1.0276	1.3728
0.30	0.1960	0.3114	0.4446	0.6023	0.7953	1.0446	1.3957
0.40	0.2023	0.3214	0.4590	0.6215	0.8201	1.0776	1.4387
0.50	0.2083	0.3312	0.4727	0.6405	0.8458	1.1107	1.4834
0.60	0.2142	0.3406	0.4868	0.6597	0.8707	1.1439	1.5283
0.80	0.2252	0.3593	0.5143	0.6971	0.9213	1.2105	1.6171
1.0	0.2355	0.3773	0.5411	0.7343	0.9712	1.2774	1.7080

7.1 Examples

- (1) Let us consider a steel tube with an inside radius $r_i = 40$ mm and an outside radius $r_o = 50$ mm.

Thus $x_w/D_o = 10/100 = 0.1$.

Moreover, $c = 500$ J/kg K; $\rho = 7850$ kg/m³; $k = 45$ W/m K; $\alpha = 100$ W/m² K. The initial temperature of the tube is equal to 20°C; in order to increase it to 220°C through the external fluid which is at 520°C.

Heating is equal to $(220 - 20)/(520 - 20) = 0.4 = 40\%$.

Biot's number is equal to $100 \times 0.01/45 = 0.02222$.

Based on Table 6, $X = 0.4549$.

The required time is therefore equal to $\theta = 0.4549 \times 500 \times 7850 \times 0.01/100 = 178.5$ s.

Figure 12 shows the behavior of the temperatures in the tube for various time periods, including of course $\theta = 178.5$ s; the abscissa shows radius r .

- (2) Let us consider a steel tube with an inside radius $r_i = 30$ mm and an outside radius $r_o = 50$ mm.

Thus, $x_w/D_o = 20/100 = 0.2$.

In addition, $c = 500$ J/kg K; $\rho = 7850$ kg/m³; $k = 45$ W/m K; $\alpha = 100$ W/m² K.

Fig. 12 Steel tube:
 $D_o = 100\text{ mm}$, $x_w = 10\text{ mm}$

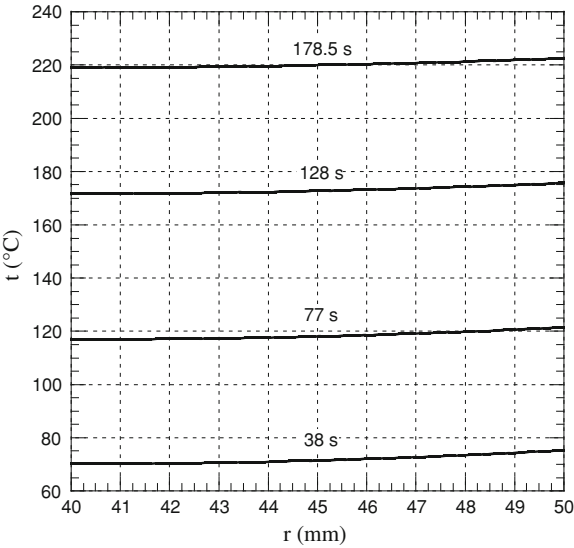
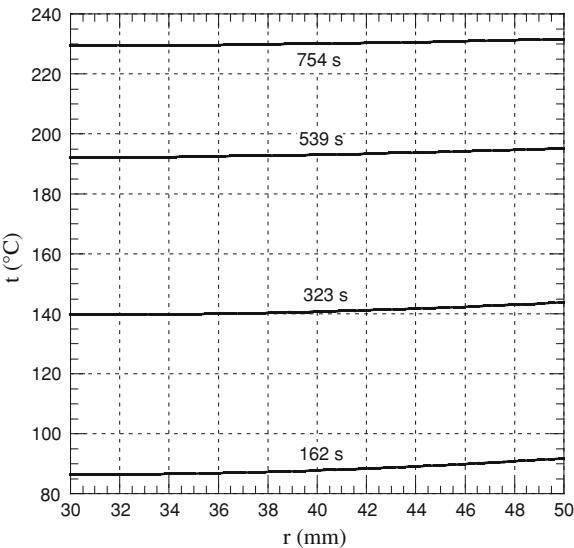
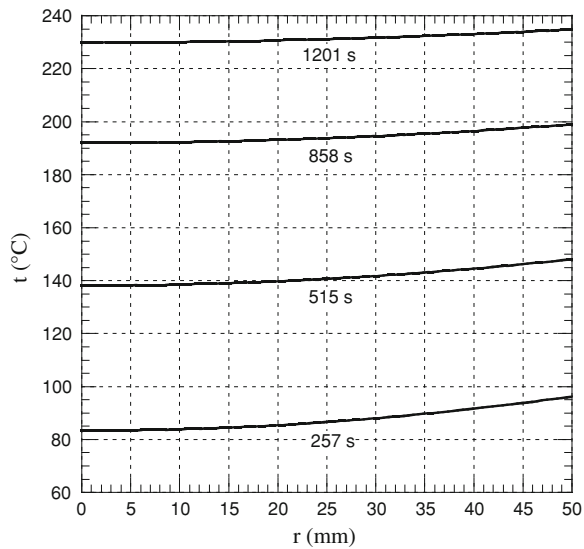


Fig. 13 Steel tube:
 $D_o = 100\text{ mm}$, $x_w = 20\text{ mm}$



The initial temperature of the tube is equal to 20°C ; it is to be raised to 230°C through licking on the outside by a fluid at 320°C .
Heating is equal to $(230 - 20)/(320 - 20) = 0.7 = 70\%$.
Biot's number is equal to $100 \times 0.02/45 = 0.04444$.
Based on Table 7, $X = 0.9603$.
The required time is equal to $\theta = 0.9603 \times 500 \times 7850 \times 0.02/100 = 754\text{ s}$.
Figure 13 shows the behavior of the temperature in the tube over various time periods, including of course $\theta = 754\text{ s}$; the abscissa shows radius r .

Fig. 14 Steel cylindrical bar:
 $D_o = 100$ mm



- (3) Let us now consider a steel bar with the same characteristics of the previous example but, of course, with $r_i = 0$ and $x_w = 50$ mm. The temperatures in question are the same.

Thus, we obtain $Bi = 100 \times 0.05/45 = 0.1111$.

Based on Table 5, $X_p = 1.2238$ and therefore based on (37), $X = 0.5 \times 1.2238 = 0.6119$.

Therefore, $\theta = 0.6119 \times 500 \times 7850 \times 0.05/100 = 1201$ s.

Figure 14 shows the behavior of temperatures for different time periods including, of course, $\theta = 1201$ s. We establish that after this time the mean temperature of the bar is very close to the expected 230°C.

8 Plate Radiated on Both Sides

We are interested in studying the time required to obtain a certain heating of the plate through radiation on both sides from a heat source at a given temperature; we assume it is represented by two walls facing each other on the sides of the plate.

Two operations are required in order to be able to use what was established for the plate immersed in fluid.

The first consists of identifying an ideal heat transfer coefficient for radiation which allows us to obtain a value of Biot's number to enter Table 4 and compute from value of X the value of θ .

The second consists of identifying the existing ratio between the surface temperature of the plate which is of interest for radiation, and the mean temperature of the wall we always referred to in previous computations.

We also observe that, given that the surface temperature of the plate varies over time, as far as radiation we will consider the mean temperature between the start and the end of the process.

Finally, the computation will have to be conducted by a trial and error process, as we shall see later on.

Let us start by considering the first problem.

As usual, if α indicates the heat transfer coefficient which is an ideal value for radiation in this case and recalling what is known about radiation:

$$\alpha(T_f - T_s) = B_m \sigma (T_f^4 - T_s^4) \quad (38)$$

then

$$\alpha = B_m \sigma (T_f + T_s) (T_f^2 + T_s^2). \quad (39)$$

In (38) and in (39) T_f is the absolute temperature of the heat source, T_s is the absolute temperature of the radiated sides of the plate, B_m is the overall black level of the radiating surfaces and of the plate sides, and σ is the constant of Stefan-Boltzmann, which is known to be equal to $5.672 \times 10^{-8} \text{ W/m}^2 \text{ K}^4$.

If we assume

$$Y = 5.672 \times 10^{-8} (T_f + T_s) (T_f^2 + T_s^2) \quad (40)$$

we may write that

$$\alpha = Y B_m. \quad (41)$$

Table 8 shows the values of Y as a function of t_f and of t_s in $^{\circ}\text{C}$.

As you see, once the values of B_m and of the temperatures t_f and t_s are known, and through Table 8 as well as (41) the value of the ideal heat transfer coefficient for radiation α can easily be computed.

At this point, though, it is necessary to identify the value of t_s .

As already pointed out, we will adopt the mean value of such temperature while considering the initial as well as the final condition of the plate; but if the initial value is well known, the final one depends on the ratio between the temperature of the radiated sides and the mean temperature of the plate which we impose in order to obtain the time required to reach it, as we did so far.

By indicating this ratio by t_s/t_m (t_m stands for the mean temperature of the plate) we establish that even this ratio, as the dimensionless number X , depends on Biot's number and on the heating percentage of the plate (see Tables 4 and 5).

Thus, it was possible to create Table 9; through t_m it allows us to obtain the required value of t_s .

Clearly, this ratio is slightly above unity for low values of Biot's number, specifically for high heating percentages, but if Biot's number is high, specifically for low heating percentages, the ratio in question can be high and therefore not negligible.

Table 8 Factor Y

t_f	t_s									
	80	100	120	140	160	180	200	220	240	260
400	33.64	35.15	36.75	38.44	40.21	42.07				
420	35.91	37.48	39.13	40.86	42.68	44.59	46.59			
440	38.30	39.92	41.61	43.39	45.27	47.23	49.28			
460	40.80	42.47	44.21	46.05	47.97	49.98	52.09	54.30		
480	43.42	45.13	46.93	48.82	50.79	52.86	55.03	57.29		
500	46.15	47.92	49.77	51.71	53.74	55.86	58.08	60.40	62.82	
520	49.01	50.83	52.73	54.72	56.81	58.99	61.26	63.64	66.12	
540	51.99	53.86	55.82	57.86	60.00	62.24	64.57	67.01	69.55	72.20
560	55.10	57.02	59.03	61.14	63.33	65.63	68.02	70.51	73.11	75.82
580	58.33	60.31	62.38	64.54	66.79	69.14	71.60	74.15	76.81	79.58
600	61.70	63.74	65.86	68.08	70.39	72.80	75.31	77.93	80.65	83.49
620	65.21	67.30	69.48	71.75	74.12	76.60	79.17	81.85	84.63	87.53
640	68.85	71.00	73.23	75.57	78.00	80.53	83.17	85.91	88.76	91.72
660	72.63	74.83	77.13	79.53	82.02	84.62	87.31	90.12	93.03	96.06
680	76.55	78.82	81.18	83.63	86.19	88.85	91.61	94.48	97.46	100.55
700	80.62	82.95	85.37	87.89	90.50	93.23	96.05	98.99	102.04	105.20
720	84.84	87.23	89.71	92.29	94.97	97.76	100.65	103.65	106.77	110.00
740	89.21	91.66	94.21	96.85	99.60	102.45	105.41	108.48	111.66	114.96
760	93.74	96.25	98.86	101.57	104.38	107.30	110.32	113.46	116.71	120.08
780	98.42	100.99	103.67	106.44	109.32	112.30	115.40	118.61	121.93	125.37
800	103.26	105.90	108.64	111.48	114.42	117.48	120.64	123.92	127.31	130.82
820	108.26	110.96	113.77	116.68	119.69	122.82	126.05	129.40	132.86	136.45
840	113.43	116.20	119.07	122.05	125.13	128.33	131.63	135.05	138.59	142.25
860	118.76	121.60	124.54	127.59	130.74	134.01	137.38	140.88	144.49	148.22
880	124.27	127.17	130.18	133.30	136.53	139.86	143.31	146.88	150.57	154.37
900	129.94	132.92	136.00	139.19	142.49	145.90	149.42	153.06	156.82	160.71
920	135.80	138.84	142.00	145.26	148.63	152.11	155.71	159.42	163.26	167.23
940	141.83	144.95	148.17	151.50	154.95	158.50	162.18	165.97	169.89	173.93
960	148.04	151.23	154.53	157.94	161.45	165.09	168.84	172.71	176.70	180.82
980	154.44	157.70	161.07	164.55	168.15	171.86	175.69	179.64	183.71	187.91
1000	161.06	164.36	167.80	171.36	175.03	178.82	182.73	186.75	190.91	195.19

t_f	t_s									
	280	300	320	340	360	380	400	420	440	460
580	82.46									
600	86.43									
620	90.54	93.67								
640	94.80	97.99								
660	99.20	102.46	105.84							
680	103.76	107.09	110.54							
700	108.47	111.87	115.39	119.03						
720	113.34	116.81	120.40	124.12						
740	118.38	121.91	125.58	129.37	133.28					

(continued)

Table 8 (continued)

t_f	t_s									
	280	300	320	340	360	380	400	420	440	460
760	123.57	127.18	130.91	134.78	138.77					
780	128.93	132.61	136.42	140.36	144.43	148.63				
800	134.46	138.21	142.10	146.11	150.26	154.54				
820	140.16	143.99	147.95	152.04	156.26	160.62	165.11			
840	146.03	149.94	153.97	158.14	162.44	166.88	171.46			
860	152.08	156.06	160.18	164.42	168.80	173.32	177.98	182.78		
880	158.31	162.37	166.56	170.89	175.35	179.94	184.69	189.57		
900	164.72	168.86	173.13	177.53	182.08	186.76	191.58	196.55	201.66	
920	171.32	175.53	179.89	184.37	188.99	193.76	198.66	203.72	208.92	
940	178.10	182.40	186.83	191.40	196.10	200.95	205.94	211.08	216.37	221.81
960	185.07	189.45	193.97	198.62	203.41	208.34	213.42	218.64	224.02	229.54
980	192.24	196.70	201.30	206.03	210.91	215.93	221.09	226.40	231.86	237.48
1000	199.60	204.15	208.83	213.65	218.61	223.71	228.96	234.36	239.92	245.62

Note that Biot's number is unknown until the value of α is known, and this depends, as we saw, on Y , i.e., on t_s ; the latter value also depends on Biot's number. This is why proceeding by trial and error is required, as we shall see through the examples. Generally, though, it suffices to proceed through two steps to obtain values of the different parameters which further steps would not significantly vary.

Once Biot's number is defined and the heating percentage is known, the value of the dimensionless number X , as well as time θ , can be obtained through Table 4.

The result is approximate; in fact, if it is true that a certain value of α and Bi lead to a certain value of X and θ through Table 4, we should not forget the fact that the value of α is obtained based on the assumption that the temperature of reference of the surface of the plate is the mean value between the temperature at the beginning of the process and the one at the end of it; this is a conventional temperature which may not correspond to the real phenomenon, given that this is radiation; nonetheless, such solution is at least the simplest one and other solutions would be difficult to identify.

At this point examples are the best way to clarify the required procedure.

8.1 Examples

- (1) Let us increase the temperature of a steel plate with a thickness of 60 mm from 20 to 460°C through radiation on both sides.

The heat source is assumed to be at 900°C and the overall black level B_m to be equal to 0.6.

Moreover, $c = 500$ J/kg K; $\rho = 7850$ kg/m³; $k = 45$ W/m K.

Table 9 Ratio t_s/t_m for plate radiated on both sides

Bi	Plate heating (%)						
	20	30	40	50	60	70	80
0.010	1.0067	1.0039	1.0025	1.0017	1.0011	1.0007	1.0004
0.015	1.0101	1.0059	1.0038	1.0025	1.0017	1.0011	1.0006
0.020	1.0134	1.0078	1.0050	1.0034	1.0022	1.0014	1.0008
0.025	1.0168	1.0098	1.0063	1.0042	1.0028	1.0018	1.0010
0.030	1.0201	1.0117	1.0075	1.0050	1.0034	1.0022	1.0013
0.040	1.0268	1.0156	1.0100	1.0067	1.0045	1.0029	1.0017
0.050	1.0334	1.0195	1.0125	1.0084	1.0056	1.0036	1.0021
0.060	1.0401	1.0234	1.0150	1.0100	1.0067	1.0043	1.0025
0.070	1.0467	1.0272	1.0175	1.0117	1.0078	1.0050	1.0029
0.080	1.0533	1.0311	1.0200	1.0133	1.0089	1.0057	1.0033
0.10	1.0664	1.0387	1.0243	1.0166	1.0111	1.0071	1.0042
0.12	1.0795	1.0464	1.0298	1.0199	1.0132	1.0085	1.0050
0.15	1.0990	1.0577	1.0371	1.0247	1.0165	1.0106	1.0062
0.20	1.1311	1.0764	1.0491	1.0328	1.0218	1.0140	1.0082
0.25	1.1627	1.0949	1.0610	1.0407	1.0271	1.0174	1.0102
0.30	1.1940	1.1191	1.0727	1.0485	1.0323	1.0208	1.0121
0.40	1.2552	1.1489	1.0957	1.0638	1.0425	1.0273	1.0159
0.50	1.3147	1.1836	1.1180	1.0787	1.0525	1.0337	1.0197
0.60	1.3727	1.2174	1.1398	1.0932	1.0621	1.0399	1.0233
0.80	1.4838	1.2824	1.1816	1.1210	1.0807	1.0519	1.0303
1.0	1.5875	1.3440	1.2212	1.1475	1.0983	1.0632	1.0369
1.2	1.6865	1.4025	1.2588	1.1725	1.1150	1.0739	1.0431
1.4	1.7784	1.4579	1.2945	1.1963	1.1309	1.0841	1.0491
1.7	1.9051	1.5355	1.3448	1.2299	1.1532	1.0985	1.0575
2.0	2.0200	1.6071	1.3913	1.2610	1.1740	1.1118	1.0652
2.5	2.1890	1.7145	1.4617	1.3081	1.2054	1.1320	1.0770
3.0	2.3352	1.8089	1.5242	1.3493	1.2333	1.1500	1.0875
4.0	2.5767	1.9666	1.6294	1.4207	1.2806	1.1804	1.1052
5.0	2.7700	2.0931	1.7146	1.4783	1.3191	1.2051	1.1197
6.0	2.9300	2.1969	1.7847	1.5257	1.3508	1.2256	1.1316
8.0	3.1819	2.3578	1.8929	1.5991	1.4000	1.2572	1.1500
10.	3.3740	2.4772	1.9727	1.6532	1.4363	1.2805	1.1636

If we momentarily ignore the difference between surface temperature and mean temperature of the plate, the mean surface temperature of the plate from beginning to end of the heating process is equal to $(20 + 460)/2 = 240^\circ\text{C}$.

Based on Table 8 with $t_f = 900^\circ\text{C}$ and $t_s = 240^\circ\text{C}$ we obtain $Y = 156.82$.

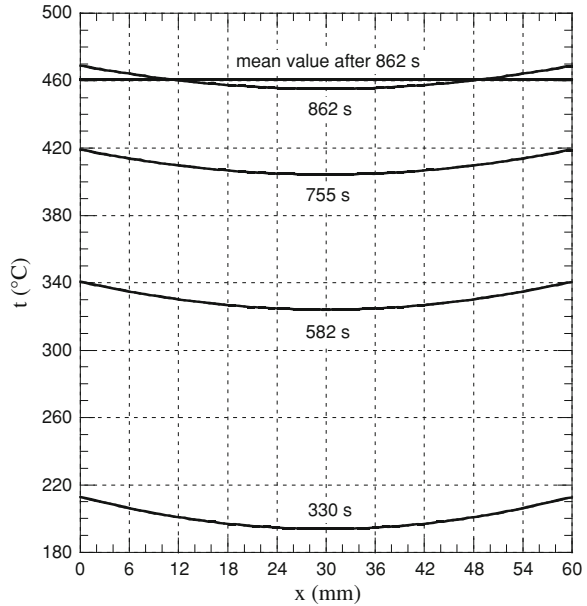
Based on (41), $\alpha = 0.6 \times 156.82 = 94.1 \text{ W/m}^2 \text{ K}$.

Biot's number is therefore equal to $94.1 \times 0.06/45 = 0.125$.

The plate heating is equal to $440/880 = 0.5 = 50\%$.

Based on Bi and the heating percentage through Table 9 we obtain that $t_s/t_m = 1.021$; note that at the end of the process the surface temperature of the plate is not equal to 460°C but to $1.021 \times 460 = 470^\circ\text{C}$ instead.

Fig. 15 Steel plate radiated on both sides—thickness 60 mm



The mean surface temperature between beginning and end of the heating is therefore not equal to 240°C , as assumed earlier, but equal to $(20 + 470)/2 = 245^{\circ}\text{C}$.

Consequently, the value of Y is equal to 157.8 and $\alpha = 94.68 \text{ W/m}^2 \text{ K}$.

Biot's number turns out to be equal to 0.126, and at this point the process by trial and error may be considered complete; further passages would not lead to significant variations in the various parameters.

Based on the value of Bi and on the heating percentage from Table 4, we obtain $X = 0.3467$.

Therefore, $\theta = 0.3467 \times 500 \times 7850 \times 0.06/94.68 = 862 \text{ s}$, and that corresponds to about 14 min.

The behavior of the temperatures for various timings including, of course, $\theta = 862 \text{ s}$, is shown in Fig. 15.

- (2) Let us assume a steel plate with a thickness of 100 mm radiated on both sides; the goal is to increase its temperature from 20 to 300°C through a heat source at 800°C ; the overall black level is equal to 0.7.

In addition, as usual $c = 500 \text{ J/kg K}$; $\rho = 7850 \text{ kg/m}^3$; $k = 45 \text{ W/m K}$.

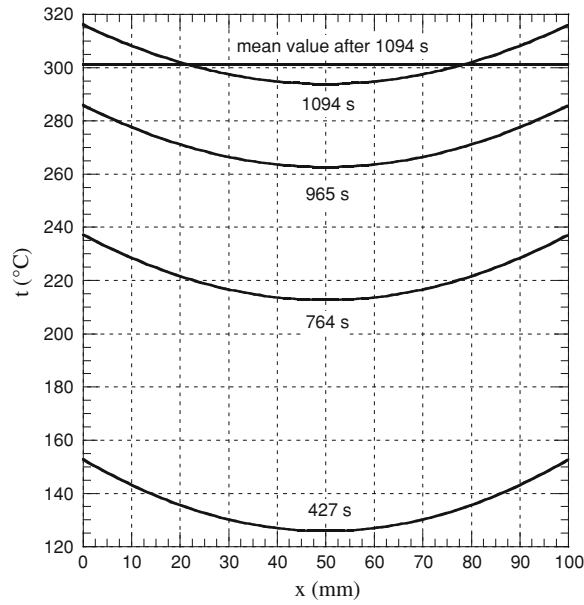
By ignoring the difference between surface temperature and mean temperature of the plate the mean surface temperature from beginning and end of the process is equal to $(20 + 300)/2 = 160^{\circ}\text{C}$.

With $t_f = 800^{\circ}\text{C}$ and $t_s = 160^{\circ}\text{C}$ through Table 8 we obtain $Y = 114.42$; thus $\alpha = 0.7 \times 114.42 = 80.1 \text{ W/m}^2 \text{ K}$ and $Bi = 80.1 \times 0.1/45 = 0.178$.

The heating process is equal to $280/780 = 0.359 = 35.9\%$.

Based on Table 9: $t_s/t_m = 1.0538$ and at the end of the process the surface temperature of the plate is equal to $1.0538 \times 300 = 316^{\circ}\text{C}$.

Fig. 16 Steel plate radiated on both sides—thickness 100 mm



The mean surface temperature is therefore equal to $(20 + 316)/2 = 168^{\circ}\text{C}$. Based on Table 8, $Y = 115.64$, thus $\alpha = 80.95 \text{ W/m}^2 \text{ K}$. Biot's number is therefore as follows: $80.95 \times 0.1/45 = 0.18$. Then, based on Table 4 $X = 0.2257$. Finally, $\theta = 0.2257 \times 500 \times 7850 \times 0.1/80.95 = 1094 \text{ s}$ which corresponds to about 18 min. The behavior of the temperatures for various timings including, of course, $\theta = 1094 \text{ s}$ is shown in Fig. 16.

9 Plate Radiated on One Side

For the plate radiated on one side and assuming no heat waste or a negligible one through the other side, the same procedure required for the plate radiated on both sides must be set in place.

Of course, the values of the ratio t_s/t_m differ from those in Table 9 for radiation on both sides; they are listed in Table 10.

As expected, the values in Table 10 are higher than those in Table 9, given the greater difficulty for the heat to penetrate the plate.

Of course, the value of Biot's number has been computed and the heating percentage is known, the value of the dimensionless number X can be obtained through Table 5; after establishing the value of X time θ may be computed, as already seen earlier.

Table 10 Ratio t_s/t_m for plate radiated on one side

Bi	Plate heating (%)						
	20	30	40	50	60	70	80
0.010	1.0133	1.0078	1.0050	1.0033	1.0022	1.0014	1.0008
0.015	1.0200	1.0116	1.0075	1.0050	1.0033	1.0021	1.0013
0.020	1.0266	1.0155	1.0100	1.0066	1.0044	1.0029	1.0017
0.025	1.0332	1.0194	1.0124	1.0083	1.0055	1.0036	1.0021
0.030	1.0398	1.0232	1.0149	1.0099	1.0066	1.0043	1.0025
0.040	1.0529	1.0309	1.0198	1.0132	1.0088	1.0057	1.0033
0.050	1.0659	1.0385	1.0247	1.0165	1.0110	1.0071	1.0041
0.060	1.0789	1.0460	1.0296	1.0197	1.0132	1.0085	1.0049
0.070	1.0918	1.0536	1.0344	1.0230	1.0153	1.0098	1.0057
0.080	1.1047	1.0610	1.0392	1.0262	1.0174	1.0112	1.0065
0.10	1.1301	1.0759	1.0488	1.0325	1.0217	1.0139	1.0081
0.12	1.1553	1.0906	1.0582	1.0388	1.0259	1.0166	1.0097
0.15	1.1926	1.1124	1.0722	1.0481	1.0321	1.0206	1.0120
0.20	1.2534	1.1478	1.0950	1.0634	1.0422	1.0272	1.0158
0.25	1.3127	1.1824	1.1173	1.0782	1.0521	1.0335	1.0195
0.30	1.3703	1.2160	1.1389	1.0926	1.0617	1.0397	1.0231
0.40	1.4809	1.2807	1.1805	1.1203	1.0802	1.0516	1.0301
0.50	1.5850	1.3421	1.2199	1.1466	1.0977	1.0628	1.0367
0.60	1.6829	1.4003	1.2574	1.1716	1.1144	1.0735	1.0429
0.80	1.8605	1.5078	1.3268	1.2179	1.1453	1.0934	1.0545
1.0	2.0163	1.6045	1.3896	1.2599	1.1732	1.1114	1.0650
1.2	2.1540	1.6915	1.4466	1.2979	1.1986	1.1277	1.0745
1.4	2.2766	1.7700	1.4983	1.3325	1.2217	1.1425	1.0831
1.7	2.4373	1.8740	1.5674	1.3789	1.2527	1.1624	1.0947
2.0	2.5764	1.9646	1.6278	1.4196	1.2799	1.1799	1.1050
2.5	2.7715	2.0916	1.7131	1.4773	1.3184	1.2047	1.1194
3.0	2.9332	2.1960	1.7834	1.5248	1.3502	1.2251	1.1313
4.0	3.1881	2.3581	1.8921	1.5985	1.3995	1.2569	1.1499
5.0	3.3823	2.4783	1.9723	1.6527	1.4358	1.2803	1.1635
6.0	3.5364	2.5716	2.0337	1.6941	1.4636	1.2981	1.1739
8.0	3.7678	2.7073	2.1219	1.7533	1.5031	1.3235	—
10	3.9342	2.8015	2.1820	1.7934	1.5298	—	—

Some examples are due at this point.

As far as the approximation of the values of θ obtained through the described procedure, please refer to what was said about the plate radiated on both sides.

9.1 Examples

- (1) Let us assume a steel plate with a thickness of 30 mm to be heated through radiation on one side from 20 to 400°C using a heat source at 1000°C; the overall black level is equal to 0.65.

Then, as usual: $c = 500 \text{ J/kg K}$; $\rho = 7850 \text{ kg/m}^3$; $k = 45 \text{ W/m K}$.

Ignoring for now the difference between the surface temperature and the mean temperature of the plate, we find that the mean surface temperature is equal to $(20 + 400)/2 = 210^\circ\text{C}$.

Based on Table 8 with $t_f = 1000^\circ\text{C}$ and $t_s = 210^\circ\text{C}$ we obtain $Y = 184.74$; then $\alpha = 0.65 \times 184.74 = 120.02 \text{ W/m}^2 \text{ K}$.

Biot's number is therefore equal to $120.02 \times 0.03/45 = 0.08$.

Heating of the plate is equal to $380/980 = 0.388 = 38.8\%$.

Based on Table 10 we obtain $t_s/t_m = 1.0418$; at the end of the process the surface temperature of the plate is therefore equal to $1.0418 \times 400 = 417^\circ\text{C}$.

The mean surface temperature is then equal to $(20 + 417)/2 = 218.5^\circ\text{C}$.

From Table 8 we obtain the new value of Y ; it is equal to 186.45; then $\alpha = 0.65 \times 186.45 = 121.19 \text{ W/m}^2 \text{ K}$.

Biot's number is therefore equal to $121.19 \times 0.03/45 = 0.0808$.

Based on this new number of Biot and on the heating percentage from Table 5 we obtain $X = 0.4954$.

Thus, $\theta = 0.4954 \times 500 \times 7850 \times 0.03/121.19 = 481 \text{ s}$, i.e., 8 min.

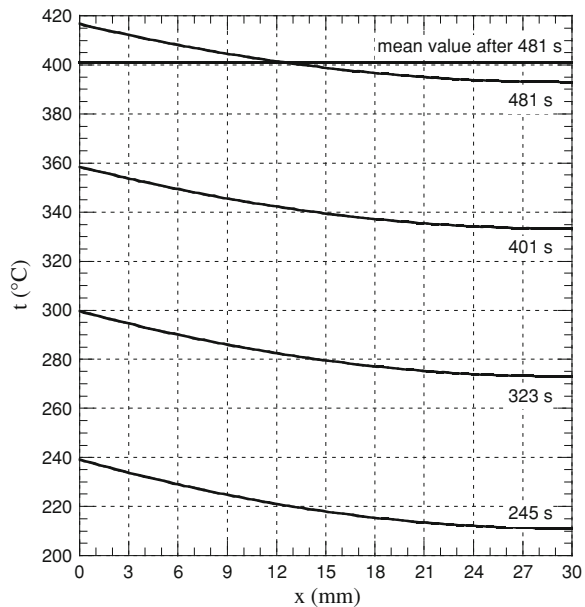
The behavior of the temperatures for various timings including of course $\theta = 481 \text{ s}$ is shown in Fig. 17.

- (2) Let us heat a plate made of refractory material with a thickness of 80 mm from 20 to 300°C through radiation from a heat source at 700°C ; 0.8 represents the value of the overall black level.

Moreover, $c = 1200 \text{ J/kg K}$; $\rho = 2000 \text{ kg/m}^3$; $k = 1.5 \text{ W/m K}$.

By first approximation the mean surface temperature of the plate is equal to $(20 + 300)/2 = 160^\circ\text{C}$.

Fig. 17 Steel plate radiated on one side—thickness 30 mm



As usual, from Table 8 we obtain the value of Y which is equal to 90.5; it follows that $\alpha = 0.8 \times 90.5 = 72.4 \text{ W/m}^2 \text{ K}$.

Biot's number is equal to $72.4 \times 0.08/1.5 = 3.86$.

The heating percentage is equal to $280/680 = 0.412 = 41.2\%$.

From Table 10 we obtain $t_s/t_m = 1.842$; this value is clearly very high and will require more passages to reach values of Bi close to actual ones.

The surface temperature of the plate at the end of the process is therefore equal to $1.842 \times 300 = 552.6^\circ\text{C}$; then the mean surface temperature is equal to $(20 + 552.6)/2 = 286.3^\circ\text{C}$.

The new value of Y is therefore equal to 109.54; thus $\alpha = 87.63 \text{ W/m}^2 \text{ K}$ and $Bi = 4.67$.

From Table 10 we obtain $t_s/t_m = 1.9084$; thus the surface temperature of the plate at the end of the process is equal to 572.5°C and the mean surface temperature is equal to 296.26°C .

The new value of Y is equal to 111.23; thus $\alpha = 88.98 \text{ W/m}^2 \text{ K}$ and $Bi = 4.74$.

From Table 10 we obtain $t_s/t_m = 1.914$; then the surface temperature of the plate at the end of the process is equal to 574.2°C and the mean surface temperature is equal to 297.1 ; then $Y = 111.38$, $\alpha = 89.1 \text{ W/m}^2 \text{ K}$ and $Bi = 4.752$.

At this point approaching the actual value of Bi can be considered complete and from Table 5 we obtain $X = 1.227$.

Therefore, $\theta = 1.227 \times 1200 \times 2000 \times 0.08/89.1 = 2644 \text{ s}$, i.e., about 44 min.

The behavior of the temperatures for various timings including of course $\theta = 2644 \text{ s}$ is shown in Fig. 18.

10 Transient Heat Transfer Through a Plane Wall

The heat transfer from a heating fluid to a heated fluid through a plate can be accomplished in a variety of different ways.

Considering transient phenomena which are of interest in this case, it is therefore impossible to elaborate a general purpose computation method to identify time θ which is required to obtain a certain result.

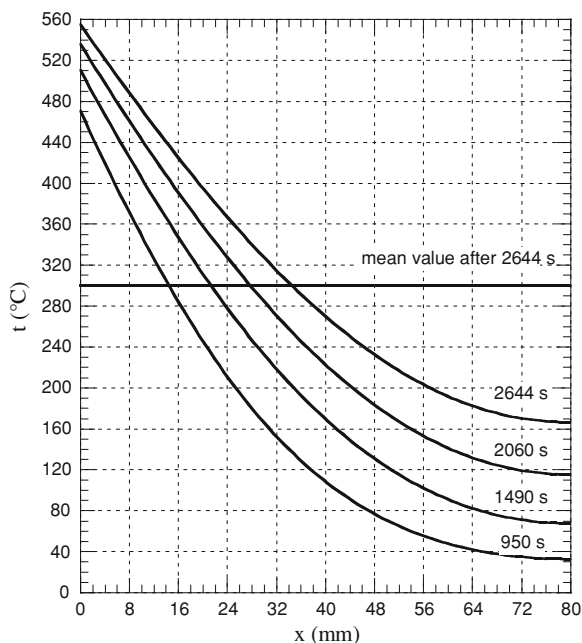
Nonetheless, after setting certain conditions it is possible to do interesting research to be, as shall be outlined next. Subsequently, we will introduce a few criteria to take into account to use the results of the research even in situations beyond the scheme at the basis of the research itself.

Let us consider a plate at an initial conventional temperature of 20°C .

The plate is licked on the sides by two fluids at different temperature, and in any case at a temperature higher than 20°C .

Initially, both fluids transfer heat to the plate, and the latter is heated up; but when the temperature of the plate surface in contact with the cooler fluid reaches

Fig. 18 Refractory plate
radiated on one side—
thickness 80 mm



the temperature of the fluid, the direction of the thermal flux is reversed, and now it is the plate transferring heat to the fluid at the expense of the heat it receives from the warmer fluid.

This leads to an almost steady condition where the distribution of the temperatures through the plate is stabilized and all the heat transferred from the warmer fluid to the plate is transferred to the cooler one.

We investigated how much time is required for 95% of the heat transferred from the warmer fluid to the plate to be transferred to the cooler one; we do not have steady condition but we are very close, i.e., only 5% of the heat transferred from the warmer fluid to the plate is still required to complete the heating process of the plate and the stabilization of temperatures.

For research to deliver simple criteria for the computation of θ it is necessary to adopt the same heat transfer coefficients α of both fluids to have just one number of Biot.

By adopting such conditions we establish that the dimensionless number X , used as usual to compute θ , essentially depends on the number of Biot as well as on the ratio between the temperatures of the two fluids indicated by t' for the heating fluid and by t'' for the heated fluid.

Thus, it was possible to create Table 11 where X is a function of Bi and of t'/t'' .

In fact, with equal values of the ratio t'/t'' , the temperature of the heating fluid (and consequently that of the heated fluid) has a certain impact on the value of X , even though this impact is modest; for instance, as far as temperature t' , going from 300 to 1000°C the increase of the value of X varies between 1 and 2%, thus making it negligible.

Table 11 Dimensionless number X for heat transfer through a plane wall with $\alpha' = \alpha''$ (warmer fluid transfers 95% of heat to colder fluid)

Bi	t'/t''									
	1.2	1.3	1.4	1.6	1.8	2.0	2.5	3	4	5
0.010	2.859	2.688	2.587	2.431	2.329	2.256	2.135	2.061	1.973	1.922
0.015	2.879	2.707	2.590	2.435	2.333	2.259	2.138	2.063	1.976	1.925
0.020	2.883	2.711	2.593	2.437	2.334	2.301	2.177	2.102	2.012	1.961
0.025	2.903	2.760	2.641	2.482	2.379	2.303	2.180	2.105	2.015	1.963
0.030	2.941	2.764	2.645	2.486	2.381	2.306	2.183	2.107	2.018	1.966
0.040	2.947	2.771	2.650	2.513	2.407	2.331	2.207	2.130	2.040	1.987
0.050	2.978	2.800	2.680	2.519	2.413	2.337	2.212	2.136	2.055	2.003
0.060	2.985	2.807	2.686	2.525	2.432	2.354	2.229	2.152	2.061	2.008
0.070	2.992	2.828	2.705	2.544	2.437	2.360	2.234	2.157	2.066	2.013
0.080	3.014	2.834	2.712	2.550	2.442	2.366	2.240	2.163	2.071	2.018
0.10	3.027	2.847	2.724	2.561	2.454	2.377	2.251	2.173	2.081	2.028
0.12	3.041	2.860	2.737	2.573	2.466	2.388	2.261	2.184	2.091	2.038
0.15	3.061	2.879	2.755	2.590	2.483	2.405	2.278	2.200	2.107	2.053
0.20	3.095	2.910	2.786	2.620	2.511	2.433	2.304	2.225	2.131	2.077
0.25	3.128	2.943	2.817	2.649	2.540	2.461	2.331	2.251	2.156	2.102
0.30	3.161	2.975	2.847	2.678	2.569	2.488	2.358	2.278	2.183	2.127
0.40	3.227	3.038	2.909	2.738	2.625	2.544	2.411	2.329	2.232	2.176
0.50	3.294	3.102	2.970	2.795	2.680	2.598	2.463	2.380	2.282	2.225
0.60	3.360	3.165	3.031	2.854	2.738	2.654	2.516	2.432	2.331	2.273
0.80	3.492	3.290	3.153	2.970	2.849	2.763	2.620	2.534	2.430	2.371
1.0	3.625	3.416	3.274	3.084	2.962	2.871	2.725	2.635	2.529	2.467
1.2	3.756	3.541	3.395	3.200	3.073	2.980	2.830	2.736	2.627	2.563
1.4	3.889	3.665	3.515	3.315	3.183	3.087	2.933	2.837	2.725	2.659
1.7	4.087	3.855	3.696	3.487	3.349	3.250	3.089	2.988	2.869	2.802
2.0	4.285	4.042	3.878	3.659	3.516	3.412	3.243	3.138	3.016	2.944

Please, note that the values of X shown in Table 11 were obtained by adopting $t' = 1000^\circ\text{C}$, and are therefore conservative.

In addition, the condition that the heat transfer coefficients of both fluids be the same as the basis of the research leading to Table 11 is actually out of the ordinary.

A more general analysis of this phenomenon must include the reasonable assumption that the heat transfer coefficients of the two fluids are different; thus, it is necessary to analyze how this condition impacts the value of X .

To that extent we start by adopting the heat transfer coefficient as a reference for the computation of the number of Biot, to be considered as far as Table 11, the average between the heat transfer coefficients of the two fluids.

The value of X obtained this way through Table 11 is indicated by X^* ;

At this point we introduce a corrective factor K so that the actual value of X is given by

$$X = KX^*. \quad (42)$$

Table 12 Corrective factor K for plate with $\alpha' > \alpha''$

α'/α''	Bi	t'/t''					α'/α''	Bi	t'/t''				
		1.2	1.8	2.5	3.5	5			1.2	1.8	2.5	3.5	5
2	0.01	1.024	1.042	1.055	1.064	1.078	6	0.01	1.128	1.176	1.218	1.244	1.273
	0.10	1.027	1.044	1.056	1.065	1.079		0.10	1.145	1.198	1.232	1.257	1.286
	0.5	1.037	1.051	1.061	1.068	1.080		0.5	1.207	1.255	1.285	1.307	1.334
	1.0	1.045	1.055	1.064	1.068	1.079		1.0	1.262	1.301	1.328	1.348	1.372
	2.0	1.051	1.058	1.063	1.065	1.073		2.0	1.319	1.346	1.366	1.380	1.400
3	0.01	1.054	1.084	1.107	1.121	1.142	8	0.01	1.156	1.223	1.263	1.296	1.334
	0.10	1.061	1.091	1.111	1.125	1.146		0.10	1.185	1.250	1.289	1.318	1.351
	0.5	1.087	1.112	1.129	1.141	1.158		0.5	1.265	1.323	1.360	1.387	1.418
	1.0	1.108	1.127	1.141	1.151	1.166		1.0	1.337	1.387	1.421	1.444	1.473
	2.0	1.126	1.139	1.148	1.153	1.165		2.0	1.416	1.453	1.480	1.498	1.523
4	0.01	1.082	1.122	1.149	1.169	1.194	10	0.01	1.186	1.262	1.306	1.338	1.374
	0.10	1.093	1.132	1.158	1.177	1.201		0.10	1.219	1.292	1.336	1.369	1.404
	0.5	1.133	1.166	1.189	1.205	1.226		0.5	1.312	1.379	1.421	1.451	1.485
	1.0	1.166	1.192	1.212	1.224	1.244		1.0	1.397	1.456	1.495	1.523	1.556
	2.0	1.197	1.215	1.229	1.237	1.251		2.0	1.497	1.543	1.574	1.595	1.624
5	0.01	1.106	1.149	1.186	1.208	1.236	12	0.01	1.212	1.293	1.342	1.377	1.415
	0.10	1.120	1.167	1.198	1.220	1.246		0.10	1.247	1.327	1.374	1.409	1.448
	0.5	1.172	1.213	1.240	1.260	1.284		0.5	1.351	1.426	1.471	1.504	1.542
	1.0	1.217	1.250	1.274	1.290	1.312		1.0	1.448	1.514	1.558	1.589	1.625
	2.0	1.261	1.284	1.301	1.312	1.330		2.0	1.564	1.617	1.653	1.678	1.710

Now we must compute the values of K ; these values combined with the values of X^* listed in Table 11, make it possible to compute the actual values of X .

By indicating the heat transfer coefficient of the heating fluid with α' and the heat transfer coefficient of the heated fluid with α'' , two distinctions are required; the first concerns the presence of condition $\alpha' > \alpha''$ and the second the presence of condition $\alpha'' > \alpha'$.

In the first case we obtain Table 12 and we establish the following: the values of K are always greater than one; they increase with an increase of α'/α'' , of Bi and of t'/t'' ; this leads to an increase of X which amounts to a few percentage points if the value of α'/α'' is not considerably greater than one but may reach or exceed 50% in certain extreme cases.

In the second case we obtain Table 13 and we establish the following: the values of K increase with an increase of α''/α' and of Bi but except for a few isolated instances they decrease with an increase of t'/t'' ; in some cases the value of K is even lower than one; the values of K are always lower than those in Table 12.

Tables 12 and 13, as well as Table 11, make it possible to compute the value of X even when the heat transfer coefficients of the two fluids are different, as we shall see in the examples.

In conclusion, we would like to point out that the assumption was an initial temperature equal to 20°C; if it is higher instead, the value of X naturally decreases; generally this is a decrease of a few percentage points and may reach

Table 13 Corrective factor K for plate with $\alpha'' > \alpha'$

α''/α'	Bi	t'/t''					α''/α'	Bi	t'/t''				
		1.2	1.8	2.5	3.5	5			1.2	1.8	2.5	3.5	5
2	0.01	1.016	1.005	0.993	0.977	0.967	6	0.01	1.112	1.104	1.081	1.047	1.015
	0.10	1.020	1.009	0.997	0.982	0.972		0.1	1.129	1.119	1.096	1.062	1.032
	0.5	1.031	1.022	1.012	0.998	0.991		0.5	1.192	1.179	1.155	1.123	1.095
	1.0	1.040	1.032	1.024	1.012	1.007		1.0	1.248	1.231	1.208	1.178	1.153
	2.0	1.048	1.042	1.036	1.027	1.024		2.0	1.306	1.288	1.267	1.241	1.222
3	0.01	1.043	1.030	1.012	0.989	0.971	8	0.01	1.139	1.137	1.120	1.083	1.047
	0.1	1.050	1.038	1.020	0.997	0.980		0.10	1.168	1.163	1.139	1.102	1.074
	0.5	1.077	1.065	1.049	1.029	1.015		0.5	1.248	1.237	1.211	1.175	1.140
	1.0	1.099	1.087	1.074	1.057	1.045		1.0	1.320	1.304	1.278	1.241	1.210
	2.0	1.120	1.109	1.098	1.085	1.078		2.0	1.401	1.379	1.354	1.322	1.296
4	0.01	1.068	1.056	1.036	1.007	0.984	10	0.01	1.168	1.170	1.154	1.115	1.076
	0.10	1.080	1.067	1.047	1.018	0.996		0.10	1.200	1.199	1.175	1.137	1.097
	0.5	1.121	1.107	1.088	1.062	1.043		0.5	1.293	1.285	1.259	1.218	1.180
	1.0	1.155	1.140	1.123	1.101	1.085		1.0	1.379	1.363	1.336	1.295	1.259
	2.0	1.189	1.175	1.168	1.142	1.131		2.0	1.479	1.457	1.427	1.390	1.358
5	0.01	1.092	1.080	1.059	1.027	0.999	12	0.01	1.193	1.199	1.184	1.145	1.103
	0.10	1.105	1.094	1.072	1.041	1.013		0.10	1.228	1.231	1.207	1.167	1.125
	0.5	1.158	1.145	1.123	1.094	1.069		0.5	1.331	1.326	1.299	1.256	1.214
	1.0	1.204	1.188	1.168	1.141	1.120		1.0	1.428	1.414	1.385	1.341	1.300
	2.0	1.251	1.234	1.216	1.194	1.179		2.0	1.545	1.521	1.489	1.448	1.411

10% if the initial temperature of the plate is equal to half the temperature of the heated fluid.

We shall not focus on this phenomenon because the number of alternatives would simply be infinite, and also because if the phenomenon is present and we ignore it we are being conservative by calculating a slightly greater time than the actual one.

10.1 Examples

- (1) Let us consider a steel plate with a thickness of 30 mm (initially at 20°C as pointed out earlier) licked on one side by a fluid at 400°C and on the other side by a fluid at 200°C; we assume the heat transfer coefficient of the fluid at 400°C to be equal to 180 W/m² K and that of the fluid at 200°C to be equal to 30 W/m² K; the mean heat transfer coefficient is therefore equal to $(180 + 30)/2 = 105$ W/m² K, so the computation is based on this value.

In addition, as usual, we have $c = 500$ J/kg K; $\rho = 7850$ kg/m³; $k = 45$ W/m K. The number of Biot is equal to $105 \times 0.03/45 = 0.07$.

The ratio between the temperatures of the fluids is equal to $400/200 = 2$.

Based on Table 11 we obtain $X^* = 2.36$.

The ratio between the heat transfer coefficients is equal to 6 with $\alpha' > \alpha''$; therefore, we must examine Table 12.

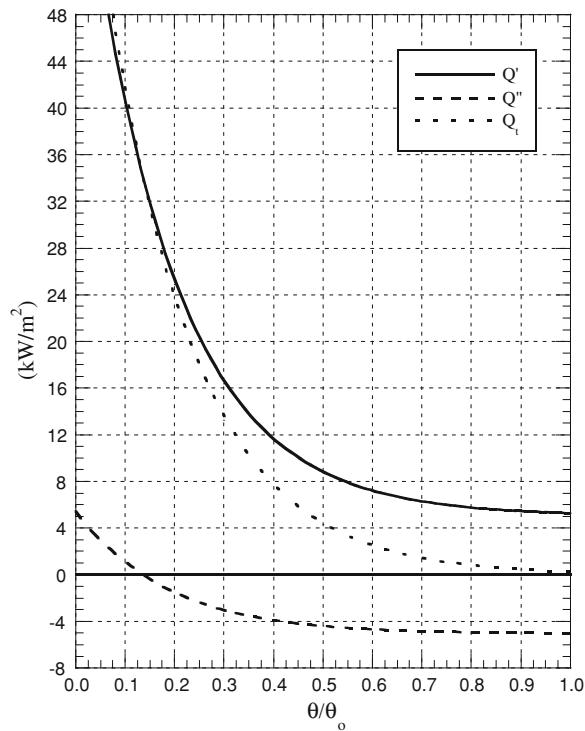
Based on the values of α'/α'' , of Bi and t'/t'' we may assume that $K = 1.2$; then $X = 1.2 \times 2.36 = 2.832$.

Thus, time is equal to $\theta = 2.832 \times 500 \times 7850 \times 0.03/105 = 3175$ s, corresponding to about 53 min, or almost 1 h.

Figure 19 shows the heat transferred from the two fluids to the plate (Q' and Q'') as a function of time, as well as the total heat stored by the plate (Q_t) assuming $\theta_0 = 3175$ s, the ratio θ/θ_0 is shown on the abscissa. Note that heat Q'' already becomes negative for values of θ slightly higher than $0.1 \theta_0$ which means that after about 350 s the plate reached the temperature of the cooler fluid, i.e., equal to 200°C ; from that moment on it the plate transfers heat to the fluid. Heat Q' decreases considerably in value as the temperature of the plate increases; the heat Q_t stored by the plate decreases considerably, as well, and is reduced to 5% of the heat transferred by the warmer fluid after 3175 s.

Direct computation with the finite differences program leads to $\theta = 3169$ s. Clearly this value practically coincides with the one computed with the suggested method; the small difference depends on the value assumed for the corrective factor K , which is not exact.

Fig. 19 Steel plate
($x_w = 30$ mm, $t' = 400^\circ\text{C}$,
 $t'' = 200^\circ\text{C}$)



If we now assume that the initial temperature of the plate to be equal to 60°C instead of 20°C , the direct computation leads to $\theta = 3100$ s with a decrease of the required time with respect to what was observed earlier to be 2.2%.

- (2) Let us assume a steel plate with a thickness of 20 mm, licked on one side by a fluid at 1000°C and on the other side by a fluid at 400°C ; the heat transfer coefficient of the fluid at 1000°C is assumed to be equal to $100 \text{ W/m}^2 \text{ K}$ whereas the heat transfer coefficient of the fluid at 400°C to be equal to $900 \text{ W/m}^2 \text{ K}$; the mean heat transfer coefficient is $500 \text{ W/m}^2 \text{ K}$; the values of c , ρ and k are the usual ones for steel.

The number of Biot is therefore equal to $500 \times 0.02/45 = 0.2222$.

The ratio between the temperatures of the fluids is equal to $1000/400 = 2.5$.

From Table 11 we obtain $X^* = 2.316$.

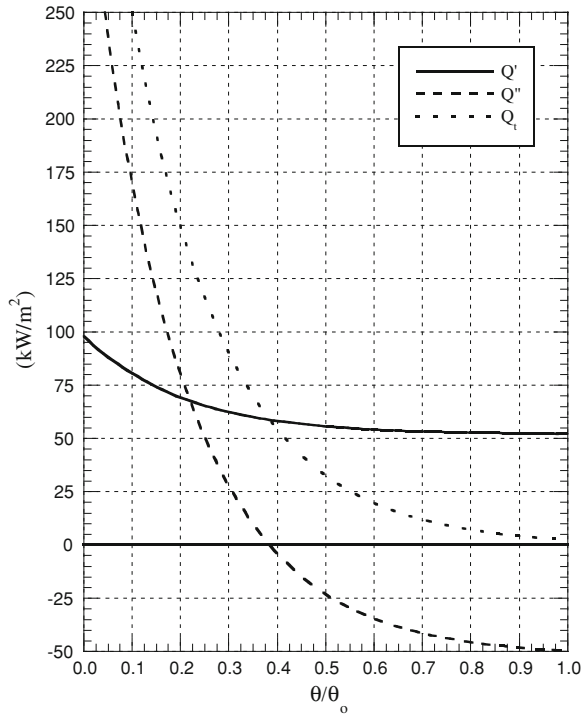
The ratio between the heat transfer coefficients is 9 and is $\alpha'' > \alpha'$, so Table 13 must be used; based on it we may assume that $K = 1.19$, and that leads to $X = 1.19 \times 2.316 = 2.756$.

Then $\theta = 2.756 \times 500 \times 7850 \times 0.02/500 = 433$ s, corresponding to about 7 min.

After 7 min 95% of the heat transferred by the warmer fluid to the wall is transferred to the cooler fluid.

Figure 20 shows the same quantities of the previous example.

Fig. 20 Steel plate
($x_w = 20$ mm, $t' = 1000^\circ\text{C}$,
 $t'' = 400^\circ\text{C}$)



Note that the value of Q' decreases over time due to the increase in temperature of the plate, even though the reduction is small given that the difference in temperature between fluid and wall remains high. The value of Q'' , which is initially very high due to the high value of the heat transfer coefficient, decreases abruptly as the temperature of the plate gets close to the one of the fluid; then the direction of the heat flux is reversed, and after about $0.4\theta_0$ the heat moves from the plate to the fluid. The initially extremely high heat Q_t stored by the plate abruptly decreases to be reduced after 433 s to 5% of the heat transferred by the warmer fluid.

Direct computation through the finite differences program leads to $\theta = 431$ s. Clearly even in this case this value practically coincides with the one computed with the suggested method.

If we now adopt an initial temperature of the plate equal to 100°C instead of 20°C , the direct computation leads to $\theta = 414$ s with a reduction of 4% with respect to the value of 431 s computed before.

11 Transient Heat Transfer Through a Curved Wall (Tube)

The typical case to be analyzed given its commonality is heat transfer through the tube wall.

Clearly, the values of the dimensionless number X relative to the tube will differ from those of the plane wall; as we shall see, such differences are not great in the case where the heat transfer coefficients have the same value for the two fluids, but they can be if these coefficients are different for the two fluids instead.

In order to allow the computation of X even for the tube we looked at the values obtained by considering that the warmer fluid to be located outside the tube, whereas the cooler one flows inside it: this is not the only possible condition but it is the most likely.

In addition, out of infinite conditions we examined the one where the ratio between the thickness of the tube and its outside diameter is equal to 0.2, i.e., $x_w/D_o = 0.2$; this may be considered restrictive but it is possible to establish that if this ratio ranges from 0 and 0.2 the value of X can be computed by interpolating between the value of X relative to the plane wall and the one relative to the tube with $x_w/D_o = 0.2$.

Table 14 shows the values of X under these conditions and assuming that heat transfer coefficients of both fluids are the same, i.e., $\alpha' = \alpha''$.

Naturally, as for the plane wall, the values of X make it possible to compute the required time for 95% of the heat transferred from the warmer fluid to be transferred to the colder fluid while 5% is stored in the tube.

A comparison of the values in Table 14 with those in Table 11 (plane wall) shows that the values of X relative to the tube are always higher than those referred to the plane wall.

Table 14 Dimensionless factor X for tube with $x_w/D_o = 0.2$ and $\alpha' = \alpha''$ (warmer fluid transfers 95% of heat to colder fluid)

Bi	t'/t''									
	1.2	1.3	1.4	1.6	1.8	2.0	2.5	3	4	5
0.010	2.899	2.732	2.619	2.471	2.390	2.322	2.212	2.146	2.069	2.025
0.015	2.917	2.751	2.636	2.489	2.392	2.325	2.214	2.148	2.071	2.028
0.020	2.920	2.754	2.642	2.491	2.397	2.328	2.223	2.189	2.110	2.065
0.025	2.926	2.779	2.690	2.537	2.441	2.371	2.260	2.191	2.112	2.069
0.030	2.979	2.809	2.692	2.542	2.443	2.374	2.262	2.194	2.114	2.071
0.040	2.985	2.814	2.699	2.547	2.471	2.399	2.286	2.217	2.138	2.093
0.050	3.018	2.845	2.730	2.575	2.477	2.404	2.291	2.224	2.143	2.099
0.060	3.024	2.850	2.735	2.580	2.482	2.424	2.309	2.240	2.159	2.114
0.070	3.033	2.859	2.754	2.601	2.499	2.429	2.314	2.245	2.165	2.119
0.080	3.054	2.878	2.760	2.605	2.505	2.433	2.321	2.251	2.170	2.124
0.10	3.066	2.890	2.775	2.618	2.517	2.446	2.330	2.261	2.180	2.133
0.12	3.082	2.906	2.785	2.631	2.530	2.456	2.341	2.272	2.189	2.143
0.15	3.100	2.923	2.805	2.647	2.545	2.475	2.358	2.288	2.206	2.158
0.20	3.135	2.956	2.836	2.676	2.575	2.502	2.383	2.313	2.232	2.184
0.25	3.169	2.989	2.865	2.708	2.605	2.530	2.412	2.339	2.258	2.210
0.30	3.202	3.021	2.898	2.738	2.632	2.557	2.437	2.367	2.281	2.235
0.40	3.270	3.084	2.959	2.795	2.689	2.614	2.492	2.419	2.332	2.285
0.50	3.336	3.149	3.021	2.855	2.745	2.670	2.545	2.470	2.383	2.333
0.60	3.403	3.212	3.084	2.914	2.803	2.724	2.597	2.522	2.435	2.382
0.80	3.538	3.339	3.207	3.030	2.917	2.836	2.706	2.626	2.535	2.482
1.00	3.670	3.467	3.329	3.146	3.030	2.946	2.809	2.728	2.635	2.580

If the heat transfer coefficients of the two fluids are different, the mean arithmetic value of the two values must be used to compute the number of Biot; a corrective factor K must be introduced by using (42), as was already done for the plane wall.

In the present case X^* is the value of X obtained through Table 14 while the value of K is obtained through Tables 15 and 16; Table 15 refers to the case where $\alpha' > \alpha''$, whereas Table 16 is to be used when $\alpha'' > \alpha'$.

As you see, the values of K can be considerably higher than one and demonstrate the impact of the ratio between the two heat transfer coefficients on the value of X and consequently on time θ .

A comparison of the values of K in Table 15 with those in Table 12 ($\alpha' > \alpha''$) establishes that the values relative to the tube are always lower than those relative to the plane wall; viceversa, a comparison of the values of K in Table 16 with those in Table 13 ($\alpha'' > \alpha'$) establishes that the values relative to the tube are always higher than those relative to the plane wall.

As already pointed out, if $x_w/D_o \neq 0.2$ the value of X for the plane wall must be computed through (42) by using Tables 11, 12 and 13; then still through (42), the value of X for the tube must be computed with $x_w/D_o = 0.2$ by using Tables 14, 15 and 16.

Table 15 Corrective factor K for tube with $x_w/D_o = 0.2$ and $\alpha' > \alpha''$

α'/α''	Bi	t'/t''					α'/α''	Bi	t'/t''				
		1.2	1.8	2.5	3.5	5			1.2	1.8	2.5	3.5	5
2	0.01	0.968	0.987	0.999	1.006	1.016	6	0.01	1.000	1.043	1.075	1.093	1.113
	0.1	0.974	0.992	1.003	1.009	1.020		0.1	1.018	1.065	1.091	1.109	1.129
	0.3	0.983	0.999	1.010	1.015	1.025		0.3	1.053	1.098	1.124	1.141	1.160
	0.6	0.995	1.009	1.019	1.023	1.033		0.6	1.099	1.141	1.166	1.182	1.201
	1.0	1.007	1.019	1.027	1.030	1.038		1.0	1.148	1.186	1.210	1.224	1.241
3	0.01	0.969	1.000	1.017	1.028	1.043	8	0.01	1.020	1.071	1.106	1.128	1.150
	0.1	0.979	1.008	1.025	1.036	1.050		0.1	1.041	1.096	1.127	1.148	1.170
	0.3	0.998	1.024	1.040	1.050	1.063		0.3	1.083	1.138	1.168	1.186	1.208
	0.6	1.021	1.045	1.059	1.067	1.081		0.6	1.140	1.190	1.219	1.238	1.260
	1.0	1.045	1.065	1.078	1.085	1.097		1.0	1.201	1.247	1.275	1.292	1.312
4	0.01	0.978	1.016	1.037	1.052	1.069	10	0.01	1.034	1.094	1.128	1.157	1.180
	0.1	0.991	1.028	1.049	1.062	1.080		0.1	1.063	1.123	1.157	1.179	1.203
	0.3	1.017	1.051	1.071	1.084	1.100		0.3	1.109	1.170	1.204	1.225	1.249
	0.6	1.050	1.081	1.099	1.110	1.125		0.6	1.173	1.231	1.264	1.284	1.309
	1.0	1.083	1.110	1.127	1.137	1.150		1.0	1.243	1.297	1.328	1.348	1.371
5	0.01	0.989	1.027	1.057	1.073	1.092	12	0.01	1.047	1.116	1.151	1.175	1.207
	0.1	1.005	1.047	1.071	1.087	1.106		0.1	1.081	1.147	1.183	1.207	1.232
	0.3	1.035	1.076	1.100	1.114	1.131		0.3	1.132	1.198	1.234	1.257	1.281
	0.6	1.076	1.113	1.134	1.148	1.166		0.6	1.201	1.265	1.300	1.323	1.349
	1.0	1.118	1.150	1.170	1.183	1.199		1.0	1.279	1.339	1.374	1.394	1.420

Table 16 Corrective factor K for tubes with $x_w/D_o = 0.2$ and $\alpha'' > \alpha'$

α''/α'	Bi	t'/t''					α''/α'	Bi	t'/t''				
		1.2	1.8	2.5	3.5	5			1.2	1.8	2.5	3.5	5
2	0.01	1.068	1.054	1.046	1.030	1.022	6	0.01	1.261	1.234	1.199	1.161	1.130
	0.1	1.079	1.061	1.048	1.033	1.026		0.1	1.286	1.255	1.220	1.180	1.146
	0.3	1.080	1.064	1.052	1.038	1.031		0.3	1.313	1.281	1.249	1.208	1.176
	0.6	1.080	1.066	1.056	1.043	1.038		0.6	1.346	1.313	1.281	1.244	1.216
	1.0	1.080	1.068	1.060	1.048	1.044		1.0	1.376	1.342	1.313	1.279	1.253
3	0.01	1.131	1.109	1.093	1.068	1.051	8	0.01	1.308	1.284	1.256	1.209	1.173
	0.1	1.146	1.120	1.099	1.075	1.059		0.1	1.344	1.319	1.281	1.235	1.193
	0.3	1.153	1.130	1.110	1.088	1.072		0.3	1.389	1.355	1.317	1.270	1.230
	0.6	1.162	1.140	1.122	1.102	1.090		0.6	1.434	1.398	1.361	1.315	1.280
	1.0	1.168	1.148	1.133	1.114	1.104		1.0	1.477	1.440	1.404	1.362	1.328
4	0.01	1.182	1.156	1.129	1.103	1.080	10	0.01	1.356	1.334	1.304	1.252	1.209
	0.1	1.201	1.171	1.144	1.113	1.091		0.1	1.396	1.373	1.332	1.281	1.233
	0.3	1.215	1.188	1.162	1.132	1.111		0.3	1.447	1.415	1.375	1.322	1.276
	0.6	1.232	1.205	1.182	1.154	1.137		0.6	1.504	1.467	1.427	1.376	1.333
	1.0	1.246	1.221	1.200	1.176	1.160		1.0	1.558	1.519	1.479	1.430	1.390
5	0.01	1.225	1.198	1.165	1.136	1.105	12	0.01	1.395	1.376	1.345	1.291	1.237
	0.1	1.247	1.216	1.185	1.149	1.120		0.1	1.438	1.418	1.375	1.322	1.269
	0.3	1.269	1.237	1.207	1.173	1.146		0.3	1.497	1.466	1.424	1.367	1.315
	0.6	1.293	1.262	1.235	1.202	1.179		0.6	1.561	1.526	1.483	1.427	1.378
	1.0	1.315	1.285	1.260	1.231	1.210		1.0	1.627	1.587	1.542	1.488	1.442

Then it is necessary to interpolate between the two identified values based on the value of x_w/D_o ; the following examples further clarify the procedure.

11.1 Examples

- (1) Let us assume a steel tube, initially at 20°C, with $r_i = 20$ mm and $r_o = 25$ mm for which then $x_w/D_o = 5/50 = 0.1$; moreover, let us assume that $t' = 250^\circ\text{C}$ and $t'' = 100^\circ\text{C}$, then $t'/t'' = 2.5$; finally, $\alpha' = 300$ W/m² K and $\alpha'' = 50$ W/m² K and then $\alpha'/\alpha'' = 6$.

The mean value of the heat transfer coefficient is therefore equal to $(300 + 50)/2 = 175$ W/m² K.

As usual, $c = 500$ J/kg K; $\rho = 7850$ kg/m³; $k = 45$ W/m K.

The number of Biot is therefore equal to $Bi = 175 \times 0.005/45 = 0.0194$.

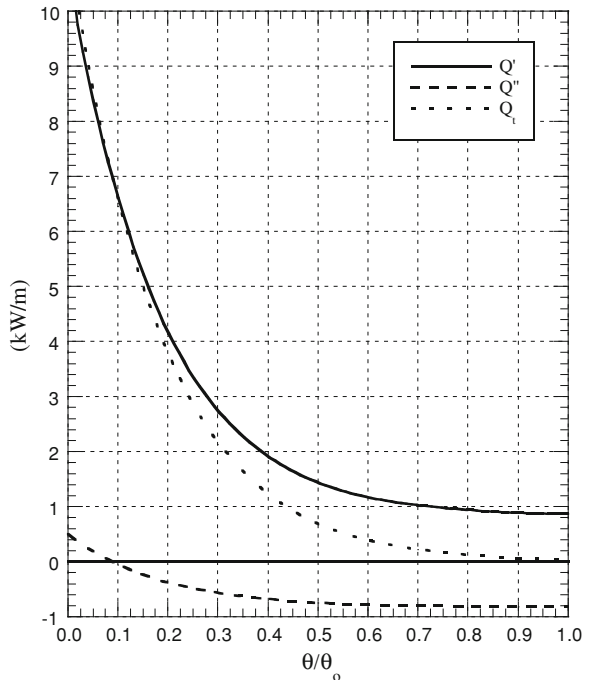
By referring to the plane wall from Table 11 we obtain $X^* = 2.172$, and from Table 12 (given that $\alpha' > \alpha''$) $K = 1.219$, so that $X = 1.219 \times 2.172 = 2.649$.

By referring to the tube instead with $x_w/D_o = 0.2$ from Table 14 we obtain $X^* = 2.222$, while from Table 15 we obtain $K = 1.078$, so that $X = 1.078 \times 2.222 = 2.395$.

Given that $x_w/D_o = 0.1$ we may assume that $X = (2.649 + 2.395)/2 = 2.522$.

Therefore, $\theta = 2.522 \times 500 \times 7850 \times 0.005/175 = 283$ s.

Fig. 21 Steel tube:
 $D_o = 50$ mm, $x_w = 5$ mm,
 $\alpha'/\alpha'' = 6$



Direct computation through the finite differences program leads to $\theta = 277$ s. As you see, our suggested computation yields a quite realistic and conservative result.

Figure 21 shows the ratio θ/θ_0 on the abscissa, given that $\theta_0 = 283$ s while the different curves are in reference to the following quantities: Q' is the heat transferred from the warmer fluid to the tube; Q'' is the heat transferred from the cooler fluid to the tube, and it becomes negative when the tube transfers heat to the colder fluid; Q_t is the heat stored by the tube.

Note that in this case the tube quickly reaches 100°C , i.e., the temperature of the colder fluid, and from that moment on the heat transfers from the tube to the fluid and is considered to be negative.

- (2) Let us assume a steel tube at an initial temperature of 20°C , with $r_i = 16$ mm and $r_o = 20$ mm then $x_w D_o = 4/40 = 0.1$; in addition, $t' = 1000^\circ\text{C}$ and $t'' = 400^\circ\text{C}$ so that $t'/t'' = 2.5$; finally, $\alpha' = 100$ W/m² K and $\alpha'' = 1000$ W/m² K; then $\alpha''/\alpha' = 10$.

As usual, $c = 500$ J/kg K; $\rho = 7850$ kg/m³; $k = 45$ W/m K.

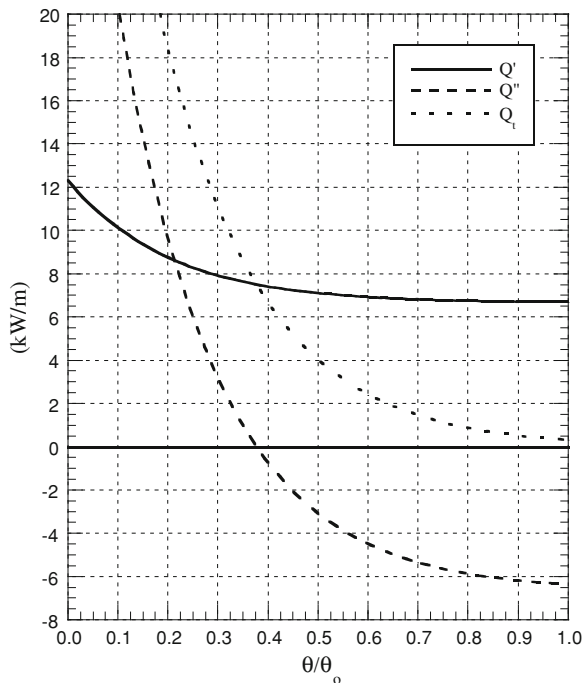
The mean value of the heat transfer coefficient is equal to $(100 + 1000)/2 = 550$ W/m² K.

The number of Biot is therefore equal to $Bi = 550 \times 0.004/45 = 0.0489$.

Considering the plane wall, from Table 11 we obtain $X^* = 2.211$, and based on Table 13 (given that $\alpha'' > \alpha'$) $K = 1.163$ so that $X = 1.163 \times 2.211 = 2.571$.

Considering the tube instead with $x_w/D_o = 0.2$, from Table 14 we obtain

Fig. 22 Steel tube:
 $D_o = 40$ mm, $x_w = 4$ mm,
 $\alpha''/\alpha' = 10$



$X^* = 2.290$, and based on Table 16 $K = 1.316$ so that $X = 1.316 \times 2.290 = 3.014$.

Given that $x_w/D_o = 0.1$ we may assume that $X = (2.571 + 3.014)/2 = 2.792$. Thus, $\theta = 2.792 \times 500 \times 7850 \times 0.004/550 = 79.7$ s.

Direct computation through the finite differences program leads to $\theta = 78.3$ s; clearly, even in this case the proposed method yields results that are quite close to actual and conservative ones.

As in the previous example, in Fig. 22 on the abscissa we show the ratio θ/θ_0 with $\theta_0 = 79.7$ s and the curves relative to Q' , Q'' and Q_r .

Contrary to the previous example, in this case the colder fluid transfers heat to the tube for a relatively long time; in fact, the temperature of the tube must reach 400°C in order to have flux inversion.